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Eight-Dimensional Spinor Representation of the Poincaré Group

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Finite-dimensional matrix representations of the Poincaré group are discussed with particular emphasis on the eight-dimensional spinor representation. It is speculated that the complex eight-dimensional representation space might be interpreted as a more fundamental entity than Minkowski space, being in a sense a square root of the latter. One can model the usual position, momentum, and angular momentum variables of a particle of nonzero rest mass and arbitrary spin by real bilinear forms in the 8-spinor components, and obtain their correct equations of motion by subjecting the spinor to a Schrödinger-like evolution equation.

1. NOTATION AND CONVENTIONS

1.1. Conjugation Operations. Superscript $*$, ^{T}, † applied to a vector or matrix denotes, respectively, the complex conjugate, transpose, Hermitian conjugate.

1.2. Designation of Groups. $O(p,q)$ and $U(p,q)$ denote the groups of $(p+q)\times(p+q)$ matrices which satisfy $O^T\eta_{pq}O=\eta_{pq}$ and $U^\dagger\eta_{pq}U=\eta_{pq}$, respectively, where η_{pq} is diagonal with p eigenvalues 1 and q eigenvalues -1. $SO(p,q)$ and $SU(p,q)$ are the subgroups whose matrices have unit determinant. $\mathbb{P}$ denotes the full Poincaré group of inhomogeneous Lorentz transformations and $\mathbb{P}_+^\uparrow$ its orthochronous proper subgroup.

1.3. Alphabet Conventions. Lower case Greek letters take the values $0,1,2,3$ and are reserved for components of Minkowski space vectors and tensors. The early lower case Latin letters $a,b,\ldots,h=0,1,2,3,5,6,7,8$ and the late letters $r,s,\ldots=0,1,2,3,5,6$ belong to $SO(2,6)$ and $SO(2,4)$ vectors

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and tensors, respectively. The letters i, j, k, l, m, n take special values specified in context. Capital Latin letters refer to spinor components, the early letters $A, B, \ldots = 1,2,3,4,5,6,7,8$ belonging to 8-spinors and the late letters $Q, R, \ldots = 1,2,3,4$ to 4-spinors.

1.4. Metric Tensors. The metrics belonging to Minkowski space, $O(2,4)$ and $O(2,6)$ are, respectively,

$$g = [g_{\kappa\lambda}] = [g^{\kappa\lambda}] = \mathrm{diag}(1,-1,-1,-1),\ \eta_6 = [\eta_{6rs}] = [\eta_6^{rs}]$$
$$= \mathrm{diag}(1,-1,-1,-1,-1,1),$$
$$\eta_8 = [\eta_{8ab}] = [\eta_8^{ab}] = \mathrm{diag}(1,-1,-1,-1,-1,1,-1,-1).$$

η_5 denotes the singular metric diag$(1,-1,-1,-1,0)$.

1.5. Levi-Citivà Tensor Densities. The permutation symbols $\varepsilon^{\iota\kappa\lambda\mu}$, ε^{rstuvw}, $\varepsilon^{abcdefgh}$ take the values $\pm 1, 0$ according as the indices form an even, odd, or not a permutation of the standard order 0123, 012356, 01235678.

1.6. Miscellaneous. If x is real $\mathrm{sgn}(x) = \pm 1, 0$ according as x is positive, negative, or zero. A bar placed over a symbol turns a column vector of contravariant components into a row vector of covariant components. Thus if a is a column vector of Minkowski vector components a^κ then $\bar{a} = a^T g$ is the row vector of components a_κ. If ψ is a 4-spinor or an 8-spinor then $\bar{\psi} = \psi^\dagger \beta$ with the appropriate β specified in context.

2. INTRODUCTION

In special relativity theory the coordinates $x^\lambda, (x^\lambda)'$, ascribed to a point event by two different inertial observers are related by a Poincaré transformation (L, a):

$$(x^\lambda)' = L^\lambda_\mu x^\mu - a^\lambda \tag{1}$$

The Lorentz matrix L is an element of the group $O(1,3)$ and thus satisfies $L^T g L = g$, while the translation $-a^\lambda$ gives the coordinates of the origin of the unprimed observer referred to the primed observer. The multiplication rule for carrying out two Poincaré transformations in sequence, (L_2, a_2) first and then (L_1, a_1), is $(L_1, a_1)(L_2, a_2) = (L_1 L_2, L_1 a_2 + a_1)$. In particular we can effect (L, a) as the homogeneous Lorentz transformation $(L,0)$ followed by the translation (I_4, a).

The full Poincaré group $\mathbb{P}$ is a nonconnected Lie group with four connected components, corresponding to the four pieces of $O(1,3)$. The component connected to the identity is a normal subgroup of $\mathbb{P}$, the orthochronous proper Poincaré group $\mathbb{P}_+^\uparrow$ whose elements preserve both the direction of time and the righthandedness of the spatial axes ($L_0^0 \geqq 1$, $\det L = 1$). If (L, a) takes all values in $\mathbb{P}_+^\uparrow$ then $(\pm Lg, a) = (L, a)(\pm g, 0)$ and $(-L, a) = (L, a)(-I_4, 0)$ range over the other three disjoint pieces of $\mathbb{P}$. $(g,0)$, $(-g,0)$, and $(-I_4,0)$ are the space inversion, time reversal, and space-time inversion operations, respectively. While these latter improper transformations cannot be effected on physical inertial observers (passive point of view) they are important in quantum theory when one seeks the parity and time reversal properties of states (active point of view). Similarly a physical observer can only undergo future-pointing timelike translations. Spacelike and past timelike translations only make sense from an active point of view.

The unitary irreducible representations of the universal covering group of $\mathbb{P}_+^\uparrow$ on a Hilbert space are well known (Wigner, 1939; Bargman, 1948; Foldy, 1956; Shirokov, 1958; Fronsdal, 1959; Lamont and Moses, 1962, 1964) and play an essential role in relativistic quantum theory. Such representations are necessarily infinitely dimensional since we are dealing with a noncompact group. On the other hand, the finite-dimensional representations are less well documented. These representations are known implicitly through the transformation properties of special relativity tensors, and through $\mathbb{P}_+^\uparrow$ being a subgroup of the conformal group, which is locally isomorphic to $SO(2,4)$ and $SU(2,2)$ (Cartan, 1914; Dirac, 1936; Murai, 1953, 1954; Bracken and Jessup, 1982). However, these finite-dimensional representations are rarely spelled out explicitly. The purpose of this paper is to make explicit some properties of these representations, with particular emphasis on the eight-dimensional spinor representation of the full Poincaré group $\mathbb{P}$. Many of these properties are already contained implicitly or in disguised form in the literature.

The plan of this paper is as follows. Section 3 summarizes the simpler real representations of $\mathbb{P}$. In Section 4 we consider the four-dimensional complex twistor representation (Dirac, 1936; Hepner, 1962; Kastrup, 1962; Penrose, 1967) expressed in a form which enables ready comparison with the results of the following section, 5. Sections 3 and 4 are not intended to be comprehensive reviews of these topics but merely to recall those aspects which help with the understanding of Section 5. The latter concerns the eight-dimensional spinor representation of $\mathbb{P}$ and contains the main results of this paper. The final section, 6, is speculative. It expands on the idea suggested in an earlier paper (Derrick, 1982) that the eight-dimensional representation space could be interpreted as a more fundamental entity than Minkowski space, being in a rough sense a square root of the latter.

3. REAL REPRESENTATIONS OF $\mathbb{P}$

In this section we consider some of the simpler representations of $\mathbb{P}$ by groups whose elements are finite matrices. Thus we seek a mapping of the elements $\pi \in \mathbb{P}$ on to $n \times n$ matrices $D^n(\pi)$ which preserves the group multiplication: $D^n(\pi_1)D^n(\pi_2) = D^n(\pi_1\pi_2)$. Only representations of dimension $n \leq 15$ will be considered here. These will be designated D_0^1, D_0^4, D^5, $\tilde{D}^5$, D_0^6, D^6, D^{10}, $\tilde{D}^{10}$, D^{15}. A subscript 0 indicates that a representation is unfaithful, i.e., the mapping $\pi \to D_0^n(\pi)$ is many to 1. A further real representation D^{20} will play a role in Section 5.

In addition to the identity representation, $D_0^1 = 1$, there are three unfaithful one-dimensional representations, $(\det L)$, $\operatorname{sgn}(L_0^0)$ and $\operatorname{sgn}(L_0^0)$ $(\det L)$. From any representation D^n of $\mathbb{P}$ we can obtain three more inequivalent representations of the same degree: $(\det L)D^n$, $\operatorname{sgn}(L_0^0)D^n$, and $\operatorname{sgn}(L_0^0)(\det L)D^n$.

3.1. The Fundamental Representation D^5. The smallest faithful representation of $\mathbb{P}$ is five-dimensional (Schweber, 1962):

$$D^5(L, a) = \begin{pmatrix} L & -a/l \\ 0 & 1 \end{pmatrix}$$

where the matrix is partitioned $4+1$ and a is the column vector of components a^κ, $\kappa = 0, 1, 2, 3$. The constant l is of dimensions length but its value has no physical significance since different values yield equivalent representations. Throughout this and subsequent sections we shall insert such a constant whenever necessary to make all representation matrix elements dimensionless.

D^5 is the representation to which the coordinates x^κ belong when supplemented by a fifth coordinate $x^4 \equiv l$. If x denotes the column vector with the five rows x^κ, x^4, then the transformation (1) takes the form $x' = D^5(L, a)x$.

Accustomed as we are to dealing with irreducible representations, it is disconcerting to note that D^5 is reducible but not completely reducible (Boerner, 1963). Maschke's theorem on complete reducibility does not apply to a noncompact group. It is characteristic of the faithful finite representations of $\mathbb{P}$ that they have invariant subspaces in which the vectors transform according to representations of $O(1,3)$. Any representation of the latter necessarily yields an unfaithful representation when extended to $\mathbb{P}$ by mapping all translations on to the unit matrix.

3.2. The Representations D_0^1 and D_0^4. From the invariant subrepresentations of D^5 we thus obtain two further representations, both unfaithful, the scalar representation $D_0^1(L, a) = 1$ and the $O(1,3)$ vector representation $D_0^4(L, a) = L$. Physical quantities which belong to these representations are, for example, the mass m of a particle and coordinate differences, both translationally invariant:

$$m' = m$$

$$(x_1^\kappa - x_2^\kappa)' = L_\mu^\kappa (x_1^\mu - x_2^\mu) \tag{2}$$

The linear momentum p^κ of a particle transforms according to $\text{sgn}(L_0^0) D_0^4$:

$$(p^\kappa)' = \text{sgn}(L_0^0) L_\mu^\kappa p^\mu \tag{3}$$

The factor $\text{sgn}(L_0^0)$ ensures that the momentum remains future pointing.

3.3. The Representations D^{10} and D_0^6. The angular momentum $j^{\iota\kappa} = -j^{\kappa\iota}$ of a particle transforms under (1) according to

$$(j^{\iota\kappa})' = \text{sgn}(L_0^0)\left[L_\lambda^\iota L_\mu^\kappa j^{\lambda\mu} - (a^\iota L_\mu^\kappa - a^\kappa L_\mu^\iota) p^\mu\right] \tag{4}$$

Combining (3) and (4) shows that p^κ and $j^{\iota\kappa}$ together belong to a ten-dimensional representation $D^{10}(L, a)$. If we write $j^{4\mu} \equiv -j^{\mu 4} = lp^\mu$, $\mu = 0,1,2,3$, then (3) and (4) show that the 5×5 matrix j^{ik}, $i, k = 0,1,2,3,4$ is an antisymmetric second-rank tensor belonging to $\text{sgn}(L_0^0) D^5 \times D^5$:

$$(j^{ik})' = \text{sgn}(L_0^0) D^{5i}{}_l D^{5k}{}_m j^{lm} \tag{5}$$

Once again we have a faithful and reducible but not completely reducible representation. As before the invariant subrepresentations correspond to translationally invariant $O(1,3)$ tensors. In addition to $\text{sgn}(L_0^0) D_0^4$ we obtain the six-dimensional representation D_0^6 associated with second-rank antisymmetric tensors, to which belongs the spin angular momentum $s^{\iota\kappa} = -s^{\kappa\iota}$:

$$(s^{\iota\kappa})' = \text{sgn}(L_0^0) L_\lambda^\iota L_\mu^\kappa s^{\lambda\mu} \tag{6}$$

3.4. The Representation D^6. There is also a faithful six-dimensional representation D^6 of $\mathbb{P}$ (Fillmore, 1977). From the space-time coordinates x^κ define $x^5 = \frac{1}{2}[l^{-1}(x_\kappa x^\kappa - K) - l]$ and $x^6 = \frac{1}{2}[l^{-1}(x_\kappa x^\kappa - K) + l]$, where K is an arbitrary constant and l a nonzero length as before. The six quantities x^r, $r = 0,1,2,3,5,6$ satisfy $\eta_{6rs}x^r x^s = K$, $\eta_{6rs}b^r x^s = l$, where $[\eta_{6rs}] = [\eta_6{}^{rs}] = \mathrm{diag}(1, -1, -1, -1, -1, 1)$ and $b^r = (0,0,0,0,1,1)$. Writing χ for the column vector with the six components x^r then (1) implies that $\chi' = D^6(L, a)\chi$ with

$$D^6(L,a) = \begin{pmatrix} L & a/l & -a/l \\ \bar{a}L/l & 1 + \frac{1}{2}\bar{a}a/l^2 & -\frac{1}{2}\bar{a}a/l^2 \\ \bar{a}L/l & \frac{1}{2}\bar{a}a/l^2 & 1 - \frac{1}{2}\bar{a}a/l^2 \end{pmatrix} \tag{7}$$

and $D^6(L, a) \in O(2,4)$. In (7) $\bar{a}$ is the row vector $a^T g$, and different values of the length l yield equivalent representations.

D^6 brings out the relation of $O(2,4)$ to $\mathbb{P}$. $O(2,4)$ is the group of 6×6 matrices D satisfying $D^T \eta_6 D = \eta_6$. If we further restrict D by $Db = b$ we obtain a subgroup of $O(2,4)$ isomorphic to $\mathbb{P}$. In the context of the conformal group one would take $K = 0$ and regard x^r as six coordinates in a projective 5-space, with $x^\lambda/(b_r x^r)$ being the Minkowski coordinates (Dirac, 1936). However, since our interest is in the Poincaré group we break the conformal invariance by giving $b_r x^r$ a constant value.

There is another, equivalent way of looking at the relation of these two groups. We can regard $\mathbb{P}$ as a degenerate case of the de Sitter groups $O(2,3), O(1,4)$ with metric $\mathrm{diag}(1, -1, -1, -1, \pm\varepsilon)$, obtained by the Inönü–Wigner contraction $\varepsilon \to 0$. (Inönü and Wigner 1953; Evans, 1967). The group of 5×5 matrices D which satisfy $D^T \eta_5 D = \eta_5$, $\det D = \pm 1$ with $\eta_5 = \mathrm{diag}(1, -1, -1, -1, 0)$ is simply the set $D^5(L, a)$, the smallest faithful representation of $\mathbb{P}$. Consider now a six-dimensional flat space with coordinates x^r, $r = 0,1,2,3,5,6$ and the metric η_6. On restricting the points to lie in the five-dimensional hyperplane $b_r x^r \equiv x^6 - x^5 = \text{const}$, implying $dx^5 = dx^6$, we obtain as effective metric the singular matrix η_5. Whence we again observe that $\mathbb{P}$ is isomorphic to the subgroup of $O(2,4)$ which leaves b invariant.

3.5. The Representation D^{15}. Consider a particle of zero rest mass with future null momentum p^λ. The equation of its null trajectory may be written $x^\lambda = z^\lambda + \tau p^\lambda$, where τ is a parameter and z^λ is any fixed point on the curve. Following Penrose (1967) we may choose z^λ as the point of intersection of the particle trajectory with the light cone through the origin, thus $z_\lambda z^\lambda = 0$. Changing coordinates according to (1), (3) yields $(x^\lambda)' = (z^\lambda)' + \tau'(p^\lambda)'$

where

$$(z^\lambda)' = L^\lambda_\mu z^\mu + \tau_0 L^\lambda_\mu p^\mu - a^\lambda$$

$$\tau' = \mathrm{sgn}(L^0_0)(\tau - \tau_0)$$

$$\tau_0 = \left(a_\lambda L^\lambda_\mu z^\mu - \tfrac{1}{2}a_\lambda a^\lambda\right)/\left(p_\iota z^\iota - a_\iota L^\iota_\kappa p^\kappa\right) \tag{8}$$

The shift in parameter by τ_0 is necessary to make $(z^\lambda)'$ null. We can allow the denominator in τ_0 to vanish if we "compactify" the Minkowski space by adding a closed null cone at infinity (Penrose, 1967). The nonlinear transformation (8) becomes linear in the 15 quantities $\xi = z_\lambda p^\lambda$,

$$\xi^\lambda = \xi z^\lambda,\ p^\lambda \quad \text{and} \quad l^{\lambda\mu} = z^\lambda p^\mu - z^\mu p^\lambda:$$

$$\xi' = \mathrm{sgn}(L^0_0)\left[\xi - a_\kappa L^\kappa_\mu p^\mu\right]$$

$$(\xi^\kappa)' = \mathrm{sgn}(L^0_0)\left[\xi^\kappa - a^\kappa \xi + a_\iota L^\iota_\lambda L^\kappa_\mu l^{\lambda\mu} + a_\iota\left(L^\iota_\lambda a^\kappa - \tfrac{1}{2}a^\iota L^\kappa_\lambda\right)p^\lambda\right] \tag{9}$$

The orbital angular momentum $l^{\lambda\mu}$ transforms according to (4) so that (3), (4), (9) together yield a faithful 15-dimensional representation D^{15}. This representation will prove of importance in the interpretation of twistors in Section 4.

D^{15} is the antisymmetric part of $\mathrm{sgn}(L^0_0)D^6 \times D^6$. This is readily seen by observing that (3), (4), (9) may be combined into

$$(l^{rs})' = \mathrm{sgn}(L^0_0)\, D^{6r}{}_t D^{6s}{}_u l^{tu} \tag{10}$$

where the antisymmetric tensor $l^{rs} = -l^{sr}$, $r, s = 0,1,2,3,5,6$ has components $l^{\lambda\mu}$, $l^{5\mu} = l^{-1}\xi^\mu - \frac{1}{2}lp^\mu$, $l^{6\mu} = l^{-1}\xi^\mu + \frac{1}{2}lp^\mu$, $l^{56} = \xi$. As with D^6, D^{15} can clearly be extended to yield a representation of the conformal group.

3.6. Other Representations. One can construct further representations from the higher-rank tensors belonging to the fundamental representation D^5. Thus $(x^\iota x^\kappa, lx^\kappa, l^2)$ together generate another faithful 15-dimensional representation, inequivalent to that of Section 3.5. It is reducible but not completely so. Among the reduced parts is a faithful 14-dimensional representation to which $(x^\iota x^\kappa - \frac{1}{4}g^{\iota\kappa}x_\lambda x^\lambda, lx^\kappa, l^2)$ belong, and an unfaithful nine-dimensional one associated with second-rank traceless symmetric $O(1,3)$ tensors. Similar considerations hold for $D^5 \times D^5 \times D^5$, etc., but since these representations are not of importance to this paper they will not be considered further.

3.7. The Inverse Transposed Representations. Given a representation $D(L,a)$ we can construct another, $\tilde{D}(L,a)=[D(L,a)^{-1}]^T$, i.e., in the representation vector space we consider covariant vectors rather than contravariant ones. Of the representations considered in Sections 3.1–3.5, D_0^1, D_0^4, D_0^6, D^6, D^{15} are equivalent to their inverse transposes, the equivalence being achieved in a trivial way by lowering contravariant tensor indices with the appropriate metric tensor, g or η_6. However, $\tilde{D}^5$ and $\tilde{D}^{10}$ are new representations, not equivalent to D^5, D^{10}. Combining (1) and (3) yields $(x^\lambda p_\lambda)'=\operatorname{sgn}(L_0^0)[x^\lambda p_\lambda - p_\mu (L^{-1})_\lambda^\mu a^\lambda]$, which shows that the five quantities $(lp_\kappa, -x^\lambda p_\lambda)$ transform according to $\operatorname{sgn}(L_0^0)\tilde{D}^5$. Similarly (1) and (6) imply that $(s_{\iota\kappa}, l^{-1}s_{\iota\lambda}x^\lambda)$ belong to $\tilde{D}^{10}$.

4. THE TWISTOR REPRESENTATION D^4

4.1. Infinitesimal Generators. In Sections 4 and 5 we will be concerned with projective representations, which, for all $\pi_1, \pi_2 \in \mathbb{P}$ satisfy $D(\pi_1)D(\pi_2)$ = (phase factor) $D(\pi_1\pi_2)$. These representations are four-valued in the sense that the phase factor will take the four values $\pm 1, \pm i$, and one can pass smoothly between any two of the values $\pm D(\pi), \pm iD(\pi)$ along suitable continuous closed paths through π in the group parameter space.

We approach the problem via the Lie algebra of $\mathbb{P}_+^\uparrow$. The ten generators are the angular momentum operator $J^{\iota\kappa}$ and the momentum operator P^κ which satisfy the commutators

$$\begin{aligned}
&[J^{\iota\kappa}, J^{\lambda\mu}] = i\hbar\left[g^{\iota\mu}J^{\kappa\lambda} + g^{\kappa\lambda}J^{\iota\mu} - g^{\iota\lambda}J^{\kappa\mu} - g^{\kappa\mu}J^{\iota\lambda}\right]\\
&[J^{\iota\kappa}, P^\lambda] = i\hbar\left[g^{\kappa\lambda}P^\iota - g^{\iota\lambda}P^\kappa\right]\\
&[P^\kappa, P^\lambda] = 0
\end{aligned} \tag{11}$$

Planck's constant appears in (11) for dimensional reasons. Given any n-dimensional matrix representation for $J^{\iota\kappa}, P^\lambda$ then $D^n(L,a)=\exp(-ia_\lambda P^\lambda/\hbar)\exp(-\frac{1}{2}i\omega_{\iota\kappa}J^{\iota\kappa}/\hbar)$ is a projective representation of $\mathbb{P}_+^\uparrow$ and a true representation of its universal covering group. The antisymmetric matrix $[\omega_{\iota\kappa}]$ parameterizes $L \in SO(1,3)$ through $L=\exp(g\omega)$. It is sufficient to consider infinitesimal values of $\omega_{\iota\kappa}$ and a^λ for which we have $D^n(L,a)=I_n - ia_\lambda P^\lambda/\hbar - \frac{1}{2}i\omega_{\iota\kappa}J^{\iota\kappa}/\hbar$. Henceforth in this paper $\omega_{\iota\kappa}$ and a^λ will be taken infinitesimal so that quadratic or higher terms may be discarded.

Consider now the relation of the Poincaré and the $SO(2,4)$ Lie algebras. The latter has 15 independent generators $M^{rs}=-M^{sr}$, $r,s=0,1,2,3,5,6$ which satisfy the commutators

$$[M^{rs}, M^{tu}] = i\left[\eta_6^{ru}M^{st} + \eta_6^{st}M^{ru} - \eta_6^{rt}M^{su} - \eta_6^{su}M^{rt}\right] \tag{12}$$

Then $\exp(-\frac{1}{2}i\Omega_{rs}M^{rs})$ gives a projective representation of the $SO(2,4)$ element $\exp(\eta_6\Omega)$, where $[\Omega_{rs}]$ is a 6×6 antisymmetric matrix. If we restrict ourselves to the Poincaré subgroup by requiring that $b^r=(0,0,0,1,1)$ be invariant then (7) implies that for infinitesimal transformations Ω_{rs} takes the values $\Omega_{\iota\kappa}=\omega_{\iota\kappa}$, $\Omega_{\kappa 5}=a_\kappa/l$, $\Omega_{\kappa 6}=-a_\kappa/l$, $\Omega_{56}=0$. Whence $\frac{1}{2}\Omega_{rs}M^{rs} = \frac{1}{2}\omega_{\iota\kappa}M^{\iota\kappa}+a_\kappa(M^{\kappa 5}-M^{\kappa 6})/l$ and we identify

$$\begin{aligned} J^{\iota\kappa} &= \hbar M^{\iota\kappa} \\ P^\kappa &= \hbar(M^{\kappa 5}-M^{\kappa 6})/l \\ &\equiv \hbar M^{\kappa 4}/l \end{aligned} \tag{13}$$

The combination 5-6 will occur so frequently in superscripts that we abbreviate it in (13) and subsequently by a superscript 4. As in the previous section the nonzero length l is inserted for dimensional consistency: M^{rs} dimensionless, $J^{\iota\kappa}$ angular momenta, P^κ linear momenta.

Hence given any matrix representation of (12) we immediately have one of (11) via (13). Upon exponentiation we obtain a projective representation of $\mathbb{P}_+^\uparrow$. One then needs to examine whether the representation can be extended to the improper Poincaré transformations.

For representations of the conformal group one supplements (13) with the generator of dilations, M^{56}, and of accelerations, $M^{5\kappa}+M^{6\kappa}$ (Mack and Salam, 1969). However, these are not needed in this paper. Bracken and Jessup (1982) give a classification of a particular series of finite representations of the Lie algebra of the group of Poincaré transformations plus dilations.

4.2. The Twistor Representation D^4. In his study of conformally invariant wave equations Dirac (1936) indirectly defined a four-dimensional projective representation D^4 of $\mathbb{P}_+^\uparrow$. The generators of D^4 were given by Hepner (1962) and Kastrup (1962), and have been widely applied in twistor theory (Penrose, 1967, 1968, 1969, 1975; Penrose and MacCallum, 1973; Qadir, 1978, 1980; Luehr and Rosenbaum, 1982) and in conformally invariant field theory (for recent reviews see Bayen, 1976; Bracken and Jessup, 1982).

Let $\sigma_1,\sigma_2,\sigma_3$, and ρ_1,ρ_2,ρ_3 be two copies of the usual Pauli spin matrices. Then the generators of D^4 may be derived from those of the four-dimensional spinor representation of $SO(2,4)$, expressed in terms of the Dirac matrices $\gamma^0=\rho_3$, $(\gamma^1,\gamma^2,\gamma^3)=i\rho_2\sigma$, $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3=-i\rho_1$:

$$\begin{aligned} M^{mn} &= \frac{i}{4}(\gamma^m\gamma^n-\gamma^n\gamma^m), \qquad m,n=0,1,2,3,5 \\ M^{6n} &= \tfrac{1}{2}\gamma^n \end{aligned} \tag{14}$$

Explicitly,

$$(M^{23}, M^{31}, M^{12}; M^{01}, M^{02}, M^{03}) = (\tfrac{1}{2}\sigma; \tfrac{1}{2}i\rho_1\sigma)$$

$$M^{\iota 5} = (\tfrac{1}{2}i\rho_2; \tfrac{1}{2}\rho_3\sigma)$$

$$M^{\iota 6} = (-\tfrac{1}{2}\rho_3; -\tfrac{1}{2}i\rho_2\sigma)$$

$$M^{\iota 4} = \tfrac{1}{2}(\rho_3 + i\rho_2)(I; \sigma),$$

$$\cdot\iota = 0,1,2,3 \qquad (15)$$

We have the identities

$$(M^{rs})^\dagger = \beta M^{rs}\beta, \qquad r,s = 0,1,2,3,5,6 \qquad (16)$$

for the real symmetric orthogonal matrix $\beta = \rho_3$.

The generators of D^4 are obtained by restriction to $\mathbb{P}_+^\uparrow$ according to the prescription (13):

$$(J^{23}, J^{31}, J^{12}; J^{01}, J^{02}, J^{03}) = \tfrac{1}{2}\hbar(\sigma; i\rho_1\sigma)$$

$$P^\kappa = \tfrac{1}{2}(\hbar/l)(\rho_3 + i\rho_2)(I; \sigma)$$

$$= \tfrac{1}{2}(\hbar/l)(I - i\gamma^5)\gamma^\kappa \qquad (17)$$

On account of (16) we have $(J^{\iota\kappa})^\dagger = \beta J^{\iota\kappa}\beta$ and $(P^\kappa)^\dagger = \beta P^\kappa \beta$ so that the representation D^4 satisfies $[(D^4)^{-1}]^\dagger = \beta D^4 \beta$. Hence the matrices of D^4 belong to $SU(2,2)$. Denoting indices of spinors belonging to D^4 by $Q, R, \ldots = 1,2,3,4$ and of those belonging to $(D^4)^*$ by $\dot{Q}, \dot{R} \ldots$ we see that the matrix elements of $\beta = \beta^{-1}$ are numerically invariant second-rank spinor components of type $\beta_{\dot{Q}R}$ or β^{QR}. Thus we can lower the contravariant index of a 4-spinor ψ^Q to get $\bar{\psi}_R = (\psi^Q)^* \beta_{\dot{Q}R}$ and form the $SO(2,4)$ scalar $\psi^\dagger \beta \psi = \bar{\psi}_R \psi^R$.

4.3. The Inverse Transposed Representation and Charge Conjugation. The representation $\tilde{D}^4 = [(D^4)^{-1}]^T = \beta(D^4)^*\beta$ is inequivalent to D^4. To see this we note that for equivalence there would need to exist a nonsingular matrix C for which $CJ^{\iota\kappa} + (J^{\iota\kappa})^T C = 0$ and $CP^\kappa + (P^\kappa)^T C = 0$. The usual value $C = \rho_1\sigma_2$ of Dirac electron theory satisfies the first relation but not the second, and hence only works for the $SO(1,3)$ subgroup. One easily shows from the irreducibility of the Pauli matrices that no suitable C exists. As a consequence of the inequivalence of D^4 and $\tilde{D}^4$ no charge conjugation operation is possible for Poincaré spinors belonging to D^4.

4.4. Space Inversion. As Penrose (1967) noted, no parity operation in the ordinary sense is possible. This can be seen as follows. If (L, a) is proper and orthochronous then so is $(gLg, ga) = (g,0)(L, a)(g,0)$. Given any representation $D(L, a)$ of $\mathbb{P}_+^\uparrow$ we can form another, $D^p(L, a) = D(gLg, ga)$. If D^p is equivalent to D then the matrix Π effecting the equivalence, $D^p = \Pi D \Pi$ (up to a phase factor), is a representation of the space inversion $(g,0)$. However in the case of D^4 we find $(D^4)^p = \rho_1\sigma_2(D^4)^*\rho_1\sigma_2$, inequivalent to D^4. Therefore we cannot extend D^4 to include space inversions. Despite the nonexistence of a charge conjugation operator $\mathscr{C}$ and of a parity operator $\mathscr{P}$ our last result suggests that a combined $\mathscr{C}\mathscr{P}$ operator is possible. In Section 5 we will show that if ψ belongs to D^4 then

$$(\mathscr{C}\mathscr{P})\psi = -\rho_1\sigma_2\psi^* \tag{18}$$

acts as the charge conjugated reflected "state." Alternatively we could represent space inversion unconventionally by the antilinear operation (18).

4.5. Time Reversal. A similar process yields an antilinear matrix operator to represent the time reversal $(-g,0)$. If (L, a) is proper and orthochronous so is $(gLg, -ga) = (-g,0)(L, a)(-g,0)$, and from a representation $D(L, a)$ of $\mathbb{P}_+^\uparrow$ we can form another, $D^t(L, a) = D(gLg, -ga)$. We find $(D^4)^t = \sigma_2(D^4)^*\sigma_2$. Hence a suitable time reversal operation on a D^4 spinor ψ is

$$\mathscr{T}\psi = -\sigma_2\psi^* \tag{19}$$

Of course arbitrary phase factors could be inserted into (18) and (19).

4.6. Real Bilinear Forms. From any D^4 spinor ψ^Q we can form 16 real linearly independent combinations of $(\psi^R)^*\psi^Q$, which we can take as the $SO(2,4)$ scalar $q = \bar{\psi}\psi$ and the $SO(2,4)$ second-rank antisymmetric tensor $m^{rs} = \bar{\psi}M^{rs}\psi$. It is interesting to classify these 16 quantities according to their transformation properties under the Poincaré subgroup. With respect to $SO(1,3)$ the tensor properties are obvious: q and m^{56} are scalars, $m^{\kappa 5}$ and $m^{\kappa 6}$ are vectors, and $m^{\iota\kappa}$ is an antisymmetric tensor of second rank. Consider now an infinitesimal translation a^λ. We transform the spinor according to $\psi' = (I_4 - ia_\lambda P^\lambda/\hbar)\psi$, $\bar{\psi}' = \bar{\psi}(I_4 + ia_\lambda P^\lambda/\hbar)$, yielding

$$q' = q$$

$$(m^{rs})' = m^{rs} - (i/l)a_\lambda\bar{\psi}\,[M^{rs}, M^{\lambda 4}]\psi \tag{20}$$

Applying (12) we find

$$(m^{\iota\kappa})' = m^{\iota\kappa} - (a^{\iota}m^{\kappa 4} - a^{\kappa}m^{\iota 4})/l$$

$$(m^{\iota 4})' = m^{\iota 4}$$

$$\tfrac{1}{2}(m^{\iota 5} + m^{\iota 6})' = \tfrac{1}{2}(m^{\iota 5} + m^{\iota 6}) + (a_{\kappa}m^{\iota\kappa} + a^{\iota}m^{56})/l$$

$$(m^{56})' = m^{56} - a_{\kappa}m^{\kappa 4}/l \tag{21}$$

Comparing (21) with (3), (4), (9) we see that the 15 components m^{rs} belong to the representation D^{15} of Section 3.5, while $m^{\iota 4}$ and $(m_{\kappa}{}^{4}, -m^{56})$ belong to $\operatorname{sgn}(L_0^0)D_0^4$ and $\operatorname{sgn}(L_0^0)\tilde{D}^5$, respectively, these being invariant subrepresentations of D^{15}. The factor $\operatorname{sgn}(L_0^0)$ is consistent with the time reversal (19).

4.7. Penrose's Null Line Interpretation. The 16 real quantities q, m^{rs} are functions of eight real parameters, the real and imaginary parts of ψ^Q. Hence they cannot be independent, but indeed satisfy the identities

$$m^{rs}m_s{}^{t} = -\tfrac{1}{4}q^2\eta_6^{rt}$$

$$m^{rs}m^{tu} + m^{st}m^{ru} + m^{tr}m^{su} = -\tfrac{1}{4}q\varepsilon^{rstuvx}m_{vx} \tag{22}$$

Relations of this type were first given by Pauli (1936) and Fierz (1937). The direct proof of (22) is straightforward but too lengthy to give here. These relations may be derived more elegantly by the methods of Campolattaro (1980), or by techniques analogous to those of Appendix B.

Taking particular index values in (22) gives $m^{\kappa 4}m_{\kappa}{}^{4} = 0$, $\xi^{\kappa}\xi_{\kappa} = 0$, $m^{56}m^{\iota\kappa} = \xi^{\iota}m^{\kappa 4} - \xi^{\kappa}m^{\iota 4} - \tfrac{1}{4}q\varepsilon^{\iota\kappa\lambda\mu}m_{\lambda\mu}$, where $\xi^{\kappa} = -\tfrac{1}{2}(m^{\kappa 5} + m^{\kappa 6})$. The two null vectors $m^{\kappa 4}$, ξ^{κ} are both future pointing, which can be seen from (15). If $m^{56} \neq 0$ the quantities $p^{\kappa} = \hbar m^{\kappa 4}/l$ and $z^{\kappa} = l\xi^{\kappa}/m^{56}$ act as the parameters of a null line according to the prescription of Section 3.5. Further the quantity $s^{\iota\kappa} = \hbar m^{\iota\kappa} - (z^{\iota}p^{\kappa} - z^{\kappa}p^{\iota}) = -\tfrac{1}{4}\hbar q\varepsilon^{\iota\kappa\lambda\mu}m_{\lambda\mu}/m^{56}$ is translationally invariant and may be interpreted as the spin angular momentum. It satisfies the identity $\tfrac{1}{2}\varepsilon_{\iota\kappa\lambda\mu}p^{\kappa}s^{\lambda\mu} = \tfrac{1}{2}\hbar q p_{\iota}$ on account of (22), so that $\tfrac{1}{2}\hbar q$ is the helicity.

5. THE EIGHT-DIMENSIONAL REPRESENTATION D^8

5.1. Motivation. There have been three main reasons for going beyond the simple twistor representation D^4 to higher dimensions.

Firstly, as we saw in Section 4, $D^4 \times (D^4)^*$ does not contain the fundamental representation D^5. It is therefore impossible to construct from a single D^4-spinor (twistor) any quantities that transform like Minkowski space coordinates x^λ. Penrose (1967) represented space-time events as the intersection of null lines, necessitating two or more independent twistors. In this regard Barut and Hauger (1973) noted the necessity for going to eight-dimensional spinors if one wanted to construct real (six-component) $O(2,4)$ vectors.

Secondly, D^4 does not allow a proper representation of the parity operator or of charge conjugation. In the context of the conformal group Murai (1958) pointed out that the simplest nontrivial spinor representation of $O(2,4)$ is eight-dimensional. For the $SO(2,4)$ subgroup this reduces to the direct sum of the two inequivalent four-dimensional representations of $SO(2,4)$ [corresponding to D^4 and $(D^4)^*$].

Finally, in order to construct a conformally invariant wave function for particles of nonzero rest mass one needs eight-component wave functions (Barut and Haugen, 1973). The four-component theories only allow a treatment of massless particles.

An analogy proves helpful here. The smallest nontrivial spinor representations of $SO(1,3)$ are two dimensional, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, which are inequivalent but related by complex conjugation. To describe the massless neutrino and antineutrino the two-component Weyl equation suffices, with a wave function belonging to $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$. This equation does not possess either space inversion or charge conjugation symmetry, but has a combined $\mathscr{CP}$ symmetry (Itzykson and Zuber, 1980). On the other hand, massive fermions need the Dirac equation, whose four components belong to the Dirac representation of $O(1,3)$, which allows both parity and charge conjugation. On restriction to $SO(1,3)$ the Dirac representation decomposes into $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$.

The representation D^8 is related in a similar way to $D^4\oplus(D^4)^*$, reducing to the latter when space inversion and charge conjugation are omitted.

5.2. The Generators of D^8. Our procedure is analogous to that in Section 4. First we find an 8×8 representation of the commutation relations (12) of $SO(2,4)$ and make the identification (13), and then we seek an extension to represent charge conjugation and the improper elements of $\mathbb{P}$.

In analogy with the derivation of the Dirac representation of $O(1,3)$, we start from the generalized Clifford algebra (Murai, 1958; Barut and Haugen, 1973) $\gamma^r\gamma^s+\gamma^s\gamma^r=2\eta_6^{rs}I$. From the general theory of Clifford algebras (Brauer and Weyl, 1935; Cartan, 1966; Salingaros, 1981, 1982) there exists an eight-dimensional representation of γ^r, unique up to an equivalence. The $SO(2,4)$ generators are then $M^{rs}=\frac{1}{4}i(\gamma^r\gamma^s-\gamma^s\gamma^r)$. The quantity $\gamma_7=-\gamma^7=\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6$ clearly has square $-I_8$ and anticommutes with all the γ^r, giving seven mutually anticommuting 8×8 matrices. We can now enlarge the algebra to that of $SO(2,6)$ (Derrick, 1982) by defining generators $M^{r7}=-M^{7r}=\frac{1}{2}i\gamma^r\gamma^7$, $M^{8r}=-M^{r8}=\frac{1}{2}i\gamma^r$, $M^{87}=-M^{78}=\frac{1}{2}i\gamma^7$. The 28 independent M^{ab}, $a,b=0,1,2,3,5,6,7,8$ satisfy the Lie algebra of $SO(2,6)$:

$$[M^{ab},M^{cd}]=i\left[\eta_8^{ad}M^{bc}+\eta_8^{bc}M^{ad}-\eta_8^{ac}M^{bd}-\eta_8^{bd}M^{ac}\right] \tag{23}$$

where $\eta_8=\mathrm{diag}[1,-1,-1,-1,-1,1,-1,-1]$.

An explicit representation for γ^r,γ^7 can be given in terms of three copies of the Pauli matrices $(\sigma_1,\sigma_2,\sigma_3)$, (ρ_1,ρ_2,ρ_3), (τ_1,τ_2,τ_3): $\gamma^0=\rho_3$, $(\gamma^1,\gamma^2,\gamma^3)=i\rho_2\boldsymbol{\sigma}$, $\gamma^5=-i\tau_1\rho_1$, $\gamma^6=\tau_2\rho_1$, $\gamma^7=i\tau_3\rho_1$. The corresponding generators of the representation D^8 of $\mathbb{P}_+^\uparrow$ given by (13) are

$$(J^{23},J^{31},J^{12};J^{01},J^{02},J^{03})=\tfrac{1}{2}\hbar(\boldsymbol{\sigma};i\rho_1\boldsymbol{\sigma})$$

$$P^\kappa=\tfrac{1}{2}(\hbar/l)(\tau_2+i\tau_1)(\rho_2;-i\rho_3\boldsymbol{\sigma}) \tag{24}$$

In Appendix A we list the 28 independent M^{ab} and the 35 independent products $M^{abcd}=M^{ab}M^{cd}=-(1/24)\varepsilon^{abcdefgh}M_{efgh}$ (a,b,c,d all different).

Given any representation M^{ab} of (23) it is clear that $(M^{ab})^\dagger$ and $(-M^{ab})^T$ are also representations. In the present case both these are equivalent to M^{ab}:

$$(M^{ab})^\dagger=\beta M^{ab}\beta$$

$$(M^{ab})^T=-CM^{ab}C \tag{25}$$

where $\beta=\tau_2\rho_2$ and $C=\tau_2\rho_1\sigma_2$ are real, orthogonal, and symmetric, and mutually anticommute. The products M^{abcd} satisfy $(M^{abcd})^\dagger=\beta M^{abcd}\beta$ and $(M^{abcd})^T=CM^{abcd}C$.

We can further extend the Lie algebra to that of $U(4,4)$ (Derrick, 1982). The generators are the 64 linearly independent 8×8 matrices I_8, M^{ab}, M^{abcd}, which are all self-adjoint with respect to β, whose eigenvalues are $1,1,1,1,-1,-1,-1,-1$. Thus, apart from the transformation needed to diagonalize β, these 64 generators yield the defining representation of $U(4,4)$. The matrices of the representation D^8 of $\mathbb{P}_+^\uparrow$ will themselves be elements of $SU(4,4)$, while the second equation of (25) implies that they are also complex orthogonal transformations in eight dimensions (up to an equivalence in each case).

Let us compare this situation with that for D^4. One obtains D^4 by restricting to the Poincaré subgroup the four-dimensional spinor representation of $SO(2,4)$, the representation matrices being elements of $SU(2,2)$. The latter is isomorphic to the covering group of $SO(2,4)$. Likewise the representation D^8 is a restriction to $\mathbb{P}_+^\uparrow$ of the eight-dimensional spinor representation of $SO(2,6)$, whose matrices belong to $SU(4,4)$. However $SU(4,4)$ is not isomorphic to the covering group of $SO(2,6)$ but contains it as a subgroup. In the case of D^4 one can attribute a physical significance to the extended representation through the relation of $SO(2,4)$ and $SU(2,2)$ to the conformal group. For D^8, however, no obvious physical significance is attached to $SO(2,6)$ and $SU(4,4)$ and to the representations of these bigger groups obtained as extensions of D^8.

5.3. Charge Conjugation. If $\pi \in \mathbb{P}_+^\uparrow$ then (25) implies the relations

$$\begin{aligned} D^8(\pi) &= \beta\left\{\left[D^8(\pi)\right]^{-1}\right\}^{\dagger}\beta \\ &= C\left\{\left[D^8(\pi)\right]^{-1}\right\}^{T}C \\ &= C\beta\left[D^8(\pi)\right]^{*}\beta C \end{aligned} \tag{26}$$

Denoting the indices of spinors belonging to D^8 by $A, B, \ldots = 1,2,3,4,5,6,7,8$ and those belonging to $(D^8)^*$ by $\dot{A}, \dot{B}, \ldots$ we see that the matrix elements of $\beta \equiv \beta^{-1} \equiv \beta^* \equiv \beta^T$ and $C \equiv C^{-1} \equiv C^* \equiv C^T$ are numerically invariant second-rank spinor components of type $\beta_{\dot{A}B} \equiv (\beta_{\dot{B}A})^*$ or $\beta^{A\dot{B}} \equiv (\beta^{B\dot{A}})^*$ and $C_{AB} \equiv C_{BA}$ or $C^{AB} \equiv C^{BA}$ or $C_{\dot{A}\dot{B}} \equiv C_{\dot{B}\dot{A}}$ or $C^{\dot{A}\dot{B}} \equiv C^{\dot{B}\dot{A}}$, respectively. Thus, for example, we can lower the contravariant index of an 8-spinor ψ^A to get the covariant $\bar{\psi}_A = (\psi^B)^*\beta_{\dot{B}A}$, and form the $SO(2,6)$ scalars $\bar{\psi}\psi = \psi^\dagger\beta\psi = \bar{\psi}_A\psi^A$ and $\psi^T C\psi = C_{AB}\psi^A\psi^B$.

According to (26) charge conjugation $\mathscr{C}$ should be defined by

$$\mathscr{C}\psi = C\beta\psi^* \tag{27}$$

or in terms of components, $(\mathscr{C}\psi)^A = C^{AB}\bar{\psi}_B$.

5.4. Representation of Space Inversion. If the space inversion $(g,0)$ is to be represented by a matrix Π then for all $\pi \in \mathbb{P}_+^\uparrow$ we must have $D^8[(g,0)\pi(g,0)] = \Pi D^8(\pi)\Pi$ up to a phase factor. In terms of the generators, Π must commute with $P^0, J^{23}, J^{31}, J^{12}$ and anticommute with $P^1, P^2, P^3, J^{01}, J^{02}, J^{03}$. By inspection of (24) either ρ_2 or $\tau_3\rho_3$ will serve. We choose $D^8(g,0) = \rho_2$, it being shown easily that the other choice yields an equivalent representation. Hence the action of the parity operator $\mathscr{P}$ on an 8-spinor ψ is defined, up to an arbitrary phase factor, as

$$\mathscr{P}\psi = \rho_2\psi \tag{28}$$

5.5. Time Reversal. The matrix $\tau_3\rho_2$ commutes with P^1, P^2, P^3, J^{23}, J^{31}, J^{12} and anticommutes with $P^0, J^{01}, J^{02}, J^{03}$. Hence for all $\pi \in \mathbb{P}_+^\uparrow$ $D^8[(-g,0)\pi(-g,0)] = \tau_3\rho_2 D^8(\pi)\tau_3\rho_2 \equiv \tau_3\rho_2 C\beta[D^8(\pi)]^*\beta C\tau_3\rho_2$. Thus $\tau_3\rho_2$ gives a representation of the time reversal $(-g,0)$. However, the operation $\psi \to \tau_3\rho_2\psi$ is not a suitable candidate for time reversal if we wish P^κ and $J^{\iota\kappa}$ to have the attributes of physical linear and angular momenta. Instead we define time reversal as the antilinear operation

$$\mathscr{T}\psi = \tau_3\rho_2\mathscr{C}\psi \tag{29}$$

with $\mathscr{C}$ as in (27). With the definition (29) the momenta transform properly:

$$\mathscr{T}(P^0, J^{01}, J^{02}, J^{03})\mathscr{T}^{-1} = (P^0, J^{01}, J^{02}, J^{03})$$

and

$$\mathscr{T}(P^1, P^2, P^3, J^{23}, J^{31}, J^{12})\mathscr{T}^{-1} = -(P^1, P^2, P^3, J^{23}, J^{31}, J^{12})$$

5.6. Relation to $D^4 \oplus (D^4)^*$. The matrix $S = \frac{1}{2}(1 + \tau_1 + \rho_1 - \tau_1\rho_1)$ is real, orthogonal, and symmetric and effects the transformation $S(\tau_1, \tau_2, \tau_3)S = (\tau_1, \tau_2\rho_1, \tau_3\rho_1)$, $S(\rho_1, \rho_2, \rho_3)S = (\rho_1, \tau_1\rho_2, \tau_1\rho_3)$. Applying this transformation to the generators of D^8 given in (24) leaves $J^{\iota\kappa}$ unchanged and transforms P^κ according to $SP^\kappa S = \frac{1}{2}(\hbar/l)[E_1(\rho_3 + i\rho_2)(I;\boldsymbol{\sigma}) - E_2(\rho_3 - i\rho_2)(I; -\boldsymbol{\sigma})]$. Here E_1 and E_2 stand for the projection operators $\frac{1}{2}(1+\tau_3)$ and $\frac{1}{2}(1-\tau_3)$ respectively, which satisfy $E_1^2 = E_1$, $E_2^2 = E_2$, $E_1E_2 = 0$, $E_1 + E_2 = I$. A further transformation by the unitary, Hermitian matrix $\Sigma = E_1 + E_2\rho_3\sigma_2$ yields

$$\Sigma S(J^{23}, J^{31}, J^{12}; J^{01}, J^{02}, J^{03})S\Sigma = \tfrac{1}{2}\hbar\left[E_1(\boldsymbol{\sigma}; i\rho_1\boldsymbol{\sigma}) - E_2(\boldsymbol{\sigma}; i\rho_1\boldsymbol{\sigma})^*\right]$$

$$\Sigma S(P^0; P^1, P^2, P^3)S\Sigma = \tfrac{1}{2}(\hbar/l)\Big[E_1(\rho_3 + i\rho_2)(I;\boldsymbol{\sigma}) - E_2(\rho_3 + i\rho_2)(I;\boldsymbol{\sigma})^*\Big] \tag{30}$$

Hence comparing (17) with (30), for $\pi \in \mathbb{P}_+^\uparrow$,

$$\Sigma S D^8(\pi) S\Sigma = E_1 D^4(\pi) + E_2\left[D^4(\pi)\right]^* \tag{31}$$

The time reversal operator defined by (29) also decomposes into a direct sum over invariant subspaces:

$$\Sigma S(\mathscr{T}\psi) = E_1(-\sigma_2)(E_1\Sigma S\psi)^* + E_2(-\sigma_2)(E_2\Sigma S\psi)^* \qquad (32)$$

which is consistent with (19). However, neither the charge conjugation operator $\mathscr{C}$ of (27) nor the parity operator $\mathscr{P}$ of (28) decomposes in this way. Nevertheless the product $\mathscr{C}\mathscr{P}$ assumes the direct sum form

$$\Sigma S(\mathscr{C}\mathscr{P}\psi) = E_1(-\rho_1\sigma_2)(E_1\Sigma S\psi)^* + E_2(-\rho_1\sigma_2)(E_2\Sigma S\psi)^* \qquad (33)$$

consistent with (18).

Summarizing the contents of (31), (32), (33), we have the reduction $D^8 = D^4 \oplus (D^4)^*$ for proper orthochronous Poincaré transformations and for $\mathscr{T}$ and $\mathscr{C}\mathscr{P}$, but not for $\mathscr{C}$ and $\mathscr{P}$ separately.

5.7. Real Bilinear Forms, $\mathbf{P}_+^\uparrow$ Behavior. From any D^8 spinor ψ^A we can form 64 real linearly independent combinations of $(\psi^B)^*\psi^A$, which we shall take as $q = \bar\psi\psi$, $m^{ab} = \bar\psi M^{ab}\psi$ and $m^{abcd} = \bar\psi M^{abcd}\psi$. There are 28 independent m^{ab}, while the identities $m^{abcd} \equiv -(1/24)\varepsilon^{abcdefgh}m_{efgh}$ reduce the number of independent m^{abcd} to 35. With respect to $SO(2,6)$, q is a scalar and m^{ab}, m^{abcd} are completely antisymmetric contravariant tensors of rank 2 and 4, respectively.

Let us classify these quantities according to their transformation properties under the Poincaré group. With respect to the homogeneous Lorentz group $SO(1,3)$ we have: eight scalars, $q, m^{56}, m^{57}, m^{58}, m^{67}, m^{68}, m^{78}, m^{0123} \equiv m^{5678}$; eight vectors $m^{\iota 5}, m^{\iota 6}, m^{\iota 7}, m^{\iota 8}, m^{\iota 678}, m^{\iota 578}, m^{\iota 568}, m^{\iota 567}$; four second-rank tensors $m^{\iota\kappa}$, $m^{\iota\kappa 58} \equiv \frac{1}{2}\varepsilon^{\iota\kappa\lambda\mu}m_{\lambda\mu}{}^{67}$, $m^{\iota\kappa 68} \equiv \frac{1}{2}\varepsilon^{\iota\kappa\lambda\mu}m_{\lambda\mu}{}^{57}$, $m^{\iota\kappa 78} \equiv \frac{1}{2}\varepsilon^{\iota\kappa\lambda\mu}m_{\lambda\mu}{}^{56}$. We need not consider the third-rank tensors because of the identifies $m^{\iota\kappa\lambda 5} \equiv \varepsilon^{\iota\kappa\lambda\mu}m_\mu{}^{678}$, $m^{\iota\kappa\lambda 6} \equiv \varepsilon^{\iota\kappa\lambda\mu}m_\mu{}^{578}$, $m^{\iota\kappa\lambda 7} \equiv \varepsilon^{\iota\kappa\lambda\mu}m_\mu{}^{568}$, $m^{\iota\kappa\lambda 8} \equiv -\varepsilon^{\iota\kappa\lambda\mu}m_\mu{}^{567}$. The combinations $m^{\iota\kappa 56} + im^{\iota\kappa 78}$, $m^{\iota\kappa 57} + im^{\iota\kappa 68}$, $m^{\iota\kappa 67} + im^{\iota\kappa 58}$ are self-dual: $m^{\iota\kappa 56} + im^{\iota\kappa 78} = \frac{1}{2}i\varepsilon^{\iota\kappa\lambda\mu}(m_{\iota\kappa}{}^{56} + im_{\iota\kappa}{}^{78})$, etc.

Consider now an infinitesimal translation a^λ, with the spinor transforming according to $\psi' = (I_8 - ia_\lambda P^\lambda/\hbar)\psi$, $\bar\psi' = \bar\psi(I_8 + ia_\lambda P^\lambda/\hbar)$:

$$q' = q$$

$$(m^{ab})' = m^{ab} - (i/l)a_\lambda\bar\psi[M^{ab}, M^{\lambda 4}]$$

$$(m^{abcd})' = m^{abcd} - (i/l)a_\lambda\bar\psi[M^{abcd}, M^{\lambda 4}] \qquad (34)$$

Applying (23) we find the translation properties of the m^{ab}:

$$
\begin{aligned}
(m^{56})' &= m^{56} - a_\kappa m^{\kappa 4}/l \\
(m^{47})' &= m^{47} \\
\tfrac{1}{2}(m^{57}+m^{67})' &= \tfrac{1}{2}(m^{57}+m^{67}) - a_\kappa m^{7\kappa}/l \\
(m^{48})' &= m^{48} \\
\tfrac{1}{2}(m^{58}+m^{68})' &= \tfrac{1}{2}(m^{58}+m^{68}) - a_\kappa m^{8\kappa}/l \\
(m^{78})' &= m^{78} \\
(m^{\iota 4})' &= m^{\iota 4} \\
\tfrac{1}{2}(m^{\iota 5}+m^{\iota 6})' &= \tfrac{1}{2}(m^{\iota 5}+m^{\iota 6}) + (a_\kappa m^{\iota\kappa} + a^\iota m^{56})/l \\
(m^{7\iota})' &= m^{7\iota} - a^\iota m^{47}/l \\
(m^{8\iota})' &= m^{8\iota} - a^\iota m^{48}/l
\end{aligned} \tag{35}
$$

In (35) we have used the linear combinations $m^{57} \pm m^{67}$, $m^{58} \pm m^{68}$, $m^{\iota 5} \pm m^{\iota 6}$ because this effects simplification. For $a, b = 0,1,2,3,5,6$ the m^{ab} behave like their D^4 analogs in (21), and thus belong to D^{15}. However, we now have two independent vectors, $(m^{7\iota}, m^{47})$ and $(m^{8\iota}, m^{48})$ belonging to the fundamental representation D^5. If $m^{47} \neq 0$ then $m^{7\iota}/m^{47}$ transforms like the coordinates of an event in Minkowski space, likewise $m^{8\iota}/m^{48}$. Further, m^{7s}, m^{8s}, $s = 0,1,2,3,5,6$ each belong to D^6 or equivalently are $SO(2,4)$ vectors, while m^{78} is translationally invariant.

The commutators $[M^{ab}, M^{cdef}]$ are given in Appendix A. Applying them to (34) yields the translation properties of m^{abcd}:

$$
\begin{aligned}
(m^{5678})' &= m^{5678} - a_\kappa m^{\kappa 478}/l \\
(m^{\iota 478})' &= m^{\iota 478} \\
\tfrac{1}{2}(m^{\iota 578}+m^{\iota 678})' &= \tfrac{1}{2}(m^{\iota 578}+m^{\iota 678}) + (a_\kappa m^{\iota\kappa 78} + a^\iota m^{5678})/l \\
(m^{\iota 567})' &= m^{\iota 567} - a_\kappa m^{\iota\kappa 47}/l \\
(m^{\iota 568})' &= m^{\iota 568} - a_\kappa m^{\iota\kappa 48}/l
\end{aligned}
$$

$$(m^{\iota\kappa 48})' = m^{\iota\kappa 48}$$

$$\tfrac{1}{2}(m^{\iota\kappa 58} + m^{\iota\kappa 68})' = \tfrac{1}{2}(m^{\iota\kappa 58} + m^{\iota\kappa 68}) - \left(a_\lambda \varepsilon^{\iota\kappa\lambda\mu} m_\mu{}^{567} + a^\iota m^{\kappa 568} - a^\kappa m^{\iota 568}\right)/l$$

$$(m^{\iota\kappa 78})' = m^{\iota\kappa 78} - (a^\iota m^{\kappa 478} - a^\kappa m^{\iota 478})/l \tag{36}$$

Once again we have used $\pm$ combinations of terms with superscripts 5,6 to achieve simplification. Further simplification results from replacing $m^{\iota\kappa 58} + m^{\iota\kappa 68} \equiv \frac{1}{2}\varepsilon^{\iota\kappa\lambda\mu}(m_{\lambda\mu}{}^{57} + m_{\lambda\mu}{}^{67})$ and $m^{\iota\kappa 48} \equiv -\frac{1}{2}\varepsilon^{\iota\kappa\lambda\mu} m_{\lambda\mu}{}^{47}$ by the self-dual $SO(1,3)$ tensors $f^{\iota\kappa} \equiv \frac{1}{2}(m^{\iota\kappa 57} + m^{\iota\kappa 67}) + \frac{1}{2}i(m^{\iota\kappa 58} + m^{\iota\kappa 68})$, $h^{\iota\kappa} = m^{\iota\kappa 47} - im^{\iota\kappa 48}$. These satisfy $f^{\iota\kappa} = \frac{1}{2}i\varepsilon^{\iota\kappa\lambda\mu} f_{\lambda\mu}$, $h^{\iota\kappa} = \frac{1}{2}i\varepsilon^{\iota\kappa\lambda\mu} h_{\lambda\mu}$, and under translations transform according to

$$(h^{\iota\kappa})' = h^{\iota\kappa}$$

$$(f^{\iota\kappa})' = f^{\iota\kappa} - \left(ia_\lambda \varepsilon^{\iota\kappa\lambda\mu} f_\mu + a^\iota f^\kappa - a^\kappa f^\iota\right)/l \tag{37}$$

$$(f^\iota)' = f^\iota - a_\kappa (h^{\iota\kappa})^*/l$$

where $f^\iota = m^{\iota 567} + im^{\iota 568}$. Thus the ten independent complex quantities $(h^{\iota\kappa})^*, f^{\iota\kappa}, f^\iota$ belong to a complex ten-dimensional representation, which we denote D_c^{10}. Alternatively if we use the real and imaginary parts rather than the self-dual combinations we have a real 20-dimensional representation $D^{20} = D_c^{10} \oplus (D_c^{10})^*$. The latter reduction of D^{20} fails when we consider the improper operations in the next section.

5.8. Real Bilinear Forms, Improper Transformations. Under the charge conjugation (27) m^{ab} is left invariant, while q and m^{abcd} change sign. This follows from the relations $(M^{ab})^T = -CM^{ab}C$, $(M^{abcd})^T = CM^{abcd}C$ and $C\beta + \beta C = 0$.

All M^{ab} and M^{abcd} either commute or anticommute with ρ_2 and the m^{ab} and m^{abcd} correspondingly are unchanged or reverse sign under the parity operation (28). We thus find the following parity properties: $q, m^{56}, m^{57}, m^{67}$ are scalars (unchanged); $m^{58}, m^{68}, m^{78}, m^{5678}$ pseudoscalars (change sign); $m^{\lambda 5}, m^{\lambda 6}, m^{\lambda 7}, m^{\lambda 567}$ vectors ($\lambda = 0$ unchanged, 1,2,3 change sign); $m^{\lambda 8}, m^{\lambda 678}, m^{\lambda 568}, m^{\lambda 578}$ pseudovectors ($\lambda = 0$ changes sign, 1,2,3 unchanged); $m^{\lambda\mu}, m^{\lambda\mu 56}, m^{\lambda\mu 57}, m^{\lambda\mu 67}$ tensors ($\lambda\mu = 23,31,12$ unchanged, 01,02,03 change sign); $m^{\lambda\mu 58}, m^{\lambda\mu 68}, m^{\lambda\mu 78}$ pseudotensors ($\lambda\mu = 23,31,12$ change sign, 01,02,03 unchanged). The self-dual combinations $f^{\iota\kappa}$ and $h^{\iota\kappa}$ transform into components of their complex conjugates, thus (f^{23}, f^{31}, f^{12})

$\rightarrow (f^{23}, f^{31}, f^{12})^*$, $(f^{01}, f^{02}, f^{03}) \rightarrow -(f^{01}, f^{02}, f^{03})^*$, $h^{\iota\kappa}$ likewise. Hence of the representations D^{20}, D_c^{10} defined following (37), only D^{20} allows a linear representation of parity.

The time reversal properties under (29) follow similarly: $q, m^{57}, m^{67}, m^{78}$ are invariant, $m^{56}, m^{58}, m^{68}, m^{5678}$ change sign; $m^{\lambda 7}$, $m^{\lambda 568}$ behave like space-time coordinates ($\lambda = 0$ changes sign, 1, 2, 3 unchanged); $m^{\lambda 5}$, $m^{\lambda 6}, m^{\lambda 8}, m^{\lambda 578}, m^{\lambda 678}$ like linear momenta ($\lambda = 0$ unchanged, 1,2,3 change sign); $m^{\lambda\mu}$, $m^{\lambda\mu 57}$, $m^{\lambda\mu 67}$, $m^{\lambda\mu 78}$ like angular momenta ($\lambda\mu = 23, 31, 12$ change sign, 01,02,03 unchanged); $m^{\lambda\mu 56}$, $m^{\lambda\mu 58}$, $m^{\lambda\mu 68}$ like duals of angular momenta ($\lambda\mu = 23, 31, 12$ unchanged, 01,02,03 change sign). The self-dual combinations transform $(f^{23}, f^{31}, f^{12}) \rightarrow -(f^{23}, f^{31}, f^{12})^*, (f^{01}, f^{02}, f^{03}) \rightarrow (f^{01}, f^{02}, f^{03})^*$, $h^{\iota\kappa}$ similarly. Thus D_c^{10} allows an antilinear representation of time reversal and D^{20} a real linear one.

5.9. Reduction of $D^8 \times (D^8)^*$. Any second rank spinor of type $\Phi^{\dot{A}B}$ can be decomposed uniquely into the sum of a Hermitian and an anti-Hermitian part: $\Phi^{\dot{A}B} = \Phi_1^{\dot{A}B} + \Phi_2^{\dot{A}B}$, where $(\Phi_1^{\dot{A}B})^* = \Phi_1^{\dot{B}A}$ and $(\Phi_2^{\dot{A}B})^* = -\Phi_2^{\dot{B}A}$. We can now replace $\Phi_k^{\dot{A}B}$, $k = 1, 2$ by linear combinations analogous to q, m^{ab}, m^{abcd} (which were formed from the Hermitian second-rank spinor $(\psi^A)^*\psi^B$):

$$\phi_k = \beta_{\dot{A}B}\Phi_k^{\dot{A}B}$$

$$\phi_k^{ab} = (\beta M^{ab})_{\dot{A}B}\Phi_k^{\dot{A}B}$$

$$\phi_k^{abcd} = (\beta M^{abcd})_{\dot{A}B}\Phi_k^{\dot{A}B} \tag{37}$$

For $k = 1$ these quantities are real and for $k = 2$ purely imaginary. Under $\mathbb{P}_+^\uparrow$ and parity their transformations behavior is just as for q, m^{ab}, m^{abcd}. For the $k = 1$ components the same is also true for charge conjugation and time reversal. These operations are defined by obvious extension of (27) and (29) as

$$(\mathscr{C}\Phi)^{\dot{A}B} = \left[(C\beta)^A{}_{\dot{E}}\Phi^{\dot{E}F}\right]^*(C\beta)^B{}_{\dot{F}}$$

$$(\mathscr{T}\Phi)^{\dot{A}B} = \left[(\tau_3\rho_2)^A{}_E\right]^*(\mathscr{C}\Phi)^{\dot{E}F}(\tau_3\rho_2)^B{}_F \tag{38}$$

The $k = 2$ components, being derived from the anti-Hermitian $\Phi_2^{\dot{A}B}$, will undergo a sign change relative to $k = 1$ components, or equivalently to q, m^{ab}, m^{abcd}, on account of the complex conjugation in (38). Thus if a subset of the components $\phi_1, \phi_1^{ab}, \phi_1^{abcd}$ belongs to a real representation D of $\mathbb{P}$ then the corresponding subset of $\phi_2, \phi_2^{ab}, \phi_2^{abcd}$ will belong to $\operatorname{sgn}(L_0^0)D$.

Collating the results of Sections 5.7 and 5.8, which pertain to the Hermitian spinor $(\psi^A)^*\psi^B$, with those of Section 3 yields the decomposition

$$\left[D^8\times(D^8)^*\right]_{\text{Hermitian}} = D_0^1 \oplus \operatorname{sgn}(L_0^0)(\det L)\oplus D^6 \oplus (\det L)D^6$$
$$\oplus D^{15}\oplus \operatorname{sgn}(L_0^0)(\det L)D^{15}\oplus D^{20} \tag{39}$$

with dimensions $64=1+1+6+6+15+15+20$. In terms of the bilinear forms, (39) may be written $(\psi^A)^*\psi^B = q\oplus m^{78}\oplus m^{7s}\oplus m^{8s}\oplus m^{rs}\oplus m^{rs78}\oplus (m^{\iota\kappa 58}, m^{\iota\kappa 68}, m^{\lambda 567}, m^{\lambda 568})$. Of course the nontrivial indecomposable parts in the reduction (39), being themselves reducible but not completely reducible, admit invariant subrepresentations. For example, D^{15} has the subrepresentations $\operatorname{sgn}(L_0^0)D_0^4$ and D^{10} to which belong $m^{\lambda 4}$ and $(m^{\lambda 4}, m^{\iota\kappa})$, respectively, while $(m^{7\lambda}, m^{47})$ and $(m^{8\lambda}, m^{48})$ belong to the subrepresentations D^5 and $(\det L)D^5$ contained in D^6 and $(\det L)D^6$.

Inserting appropriate factors $\operatorname{sgn}(L_0^0)$ into (39) yields the decomposition

$$\left[D^8\times(D^8)^*\right]_{\text{anti-Hermitian}}$$
$$= \operatorname{sgn}(L_0^0)\oplus(\det L)\oplus \operatorname{sgn}(L_0^0)D^6\oplus \operatorname{sgn}(L_0^0)(\det L)D^6$$
$$\oplus \operatorname{sgn}(L_0^0)D^{15}\oplus(\det L)D^{15}\oplus \operatorname{sgn}(L_0^0)D^{20} \tag{40}$$

Taken together (39) and (40) give the full reduction of $D^8\times(D^8)^*$.

5.10. $D^8\times D^8$, Symmetric Part. Any second-rank spinor Φ^{AB} can be split uniquely into the sum of two parts, one symmetric in A and B and the other antisymmetric. The symmetric part will transform like the products $\psi^A\psi^B$ of the components of a D^8 spinor ψ. We can form 36 linearly independent linear combinations of these products, which we shall take as $\zeta=\psi^T C\psi$ and $\zeta^{abcd}=\psi^T CM^{abcd}\psi \equiv -(1/24)\varepsilon^{abcdefgh}\zeta_{efgh}$. (Note that C and CM^{abcd} are symmetric matrices while the CM^{ab} are antisymmetric.) Let us now classify these 36 complex quantities according to their transformation properties under the Poincaré group. With respect to proper orthochronous transformations the behavior is entirely analogous to that of Section 5.7 with $q\to\zeta$ and $m^{abcd}\to\zeta^{abcd}$. Under the charge conjugation (27) $\zeta\to-\zeta^*$ and $\zeta^{abcd}\to-(\zeta^{abcd})^*$, so that the real parts change sign while the imaginary parts are invariant. The parity behavior is again as in Section 5.8 with $q\to\zeta$, $m^{abcd}\to\zeta^{abcd}$. The time reversal (29) replaces ζ by ζ^*, and ζ^{abcd} by $\pm(\zeta^{abcd})^*$ according as $\tau_3\rho_2$ commutes or anticommutes with M^{abcd}.

Collating all these results we find that the real and imaginary parts of ζ, ζ^{abcd} belong to the real representations of $\mathbb{P}$ as follows ($r,s=0,1,2,3,5,6$; $\iota,\kappa=0,1,2,3$):

$\mathrm{Re}(\zeta)\to D_0^1$, $\mathrm{Im}(\zeta)\to \mathrm{sgn}(L_0^0)$, $\mathrm{Re}(\zeta^{rs78})\to \mathrm{sgn}(L_0^0)(\det L)D^{15}$; $\mathrm{Im}(\zeta^{rs78})\to(\det L)D^{15}$; $\mathrm{Re}(\zeta^{\iota\kappa58},\zeta^{\iota\kappa68},\zeta^{\iota567},\zeta^{\iota568})\to D^{20}$, $\mathrm{Im}(\zeta^{\iota\kappa58},\zeta^{\iota\kappa68},\zeta^{\iota567},\zeta^{\iota568})\to \mathrm{sgn}(L_0^0)D^{20}$. Note that $\mathrm{Re}(\zeta^{abcd})$ behaves like m^{abcd} (Section 5.9), consistent with the identity $(\mathscr{C}\psi)^T CM^{abcd}\psi\equiv\bar{\psi}M^{abcd}\psi$.

Hence we have the resolution

$$(D^8\times D^8)_{\text{symmetric}}=D_0^1\oplus\mathrm{sgn}(L_0^0)\oplus(\det L)D^{15}\oplus\mathrm{sgn}(L_0^0)(\det L)D^{15}\oplus D^{20}\oplus\mathrm{sgn}(L_0^0)D^{20}\quad(41)$$

5.11. $D^8\times D^8$, Antisymmetric Part. Consider a second-rank antisymmetric spinor $\Phi^{AB}=-\Phi^{BA}$. Such a spinor cannot be formed (without complex conjugation) from the components of a single D^8 spinor ψ^A, but could, for example, be obtained as the antisymmetric product of two linearly independent spinors $\psi_1^A,\psi_2^A:\Phi^{AB}=\psi_1^A\psi_2^B-\psi_1^B\psi_2^A$. We can achieve reduction of the antisymmetric part of the direct product $D^8\times D^8$ by replacing Φ^{AB} by the 28 linearly independent combinations $\phi^{ab}=(CM^{ab})_{AB}\Phi^{AB}$, $a,b=0,1,2,3,5,6,7,8$.

The transformation properties of ϕ^{ab} may be found by the techniques of Sections 5.7–5.10. Under $\mathbb{P}_+^\uparrow$ and parity ϕ^{ab} behaves just like m^{ab}. The charge conjugation $(\mathscr{C}\Phi)^{AB}=(C\beta)^A{}_{\dot{E}}(C\beta)^B{}_{\dot{F}}(\Phi^{EF})^*$ replaces ϕ^{ab} by $(\phi^{ab})^*$. Time reversal $(\mathscr{T}\Phi)^{AB}=(\tau_3\rho_2)^A{}_E(\tau_3\rho_2)^B{}_F(\mathscr{C}\Phi)^{EF}$ transforms ϕ^{ab} into $\pm(\phi^{ab})^*$ depending on whether $\tau_3\rho_2$ anticommutes or commutes with M^{ab}.

The correspondence of the real and imaginary parts of ϕ^{ab} to the real representations of $\mathbb{P}$ is ($r,s=0,1,2,3,5,6$): $\mathrm{Re}(\phi^{78})\to\mathrm{sgn}(L_0^0)(\det L)$, $\mathrm{Im}(\phi^{78})\to(\det L)$; $\mathrm{Re}(\phi^{7s})\to D^6$, $\mathrm{Im}(\phi^{7s})\to\mathrm{sgn}(L_0^0)D^6$; $\mathrm{Re}(\phi^{8s})\to(\det L)D^6$, $\mathrm{Im}(\phi^{8s})\to\mathrm{sgn}(L_0^0)(\det L)D^6$; $\mathrm{Re}(\phi^{rs})\to D^{15}$, $\mathrm{Im}(\phi^{rs})\to\mathrm{sgn}(L_0^0)D^{15}$. As a check we note that $\mathrm{Re}(\phi^{ab})$ behaves exactly as m^{ab} (Section 5.9). That this should be so follows from the identity $(CM^{ab})_{AB}(\mathscr{C}\psi)^A\psi^B\equiv\bar{\psi}M^{ab}\psi$.

Thus we obtain the decomposition

$$\begin{aligned}(D^8\times D^8)_{\text{antisymmetric}}\\ &=(\det L)\oplus\mathrm{sgn}(L_0^0)(\det L)\oplus D^6\oplus\mathrm{sgn}(L_0^0)D^6\\ &\quad\oplus(\det L)D^6\oplus\mathrm{sgn}(L_0^0)(\det L)D^6\oplus D^{15}\oplus\mathrm{sgn}(L_0^0)D^{15}\end{aligned}\quad(42)$$

Combining (41) and (42) now gives the complete resolution of $D^8 \times D^8$ into indecomposable representations. We note that (39), (40), (41), (42) taken together indicate that $D^8 \times (D^8)^*$ and $D^8 \times D^8$ have the same reduction. This is a consequence of the equivalence of $(D^8)^*$ and D^8 expressed by (26). The detailed breakup, Hermitian, and anti-Hermitian for $D^8 \times (D^8)^*$, symmetric and antisymmetric for $D^8 \times D^8$, is of course different.

5.12. Identities of Pauli–Fierz Type. Here we give the D^8 analogs of the identities (22). A D^8 spinor is specified by 16 real parameters, the real and imaginary parts of ψ^A. We can form 136 linearly independent real linear combinations of the products $\psi^A\psi^B, (\psi^A\psi^B)^*, (\psi^A)^*\psi^B$ which we take as $q = \bar{\psi}\psi$, $m^{ab} = \bar{\psi}M^{ab}\psi$, $m^{abcd} = \bar{\psi}M^{abcd}\psi$ and the real and imaginary parts of $\zeta = \psi^T C\psi$ and $\zeta^{abcd} = \psi^T CM^{abcd}\psi$. These are the combinations introduced in Sections 5.7–5.11 to achieve reduction of $D^8 \times (D^8)^*$ and $D^8 \times D^8$.

Since we have 136 (quadratic) functions of 16 dependent variables we expect them to be subject to 120 independent constraints which originate from identities like $(\psi^A\psi^B)(\psi^E\psi^F) \equiv (\psi^A\psi^E)(\psi^B\psi^F)$ and $[(\psi^A)^*\psi^B][(\psi^E)^*\psi^F] \equiv [(\psi^A)^*\psi^F][(\psi^E)^*\psi^B] \equiv (\psi^A\psi^E)^*(\psi^B\psi^F)$. In Appendix B we derive the following identities:

$$m^{ae}m_e{}^b = -\tfrac{1}{4}(q^2 + \zeta^*\zeta)\eta_8^{ab} \tag{43}$$

$$\begin{aligned} m^{ab}m^{cd} + m^{bc}m^{ad} + m^{ca}m^{bd} &= -\tfrac{1}{8}\varepsilon^{abcdefgh}m_{ef}m_{gh} \\ &= qm^{abcd} + \mathrm{Re}(\zeta^*\zeta^{abcd}) \end{aligned} \tag{44}$$

$$\begin{aligned} m^{ae}m_e{}^{bcd} &= -\tfrac{1}{4}q\left(\eta_8^{ab}m^{cd} + \eta_8^{ac}m^{db} + \eta_8^{ad}m^{bc}\right) \\ &\quad -\tfrac{1}{2}\mathrm{Im}(\zeta^*\zeta^{abcd}) \end{aligned} \tag{45}$$

$$\begin{aligned} m^{abcg}m^{def}{}_g &= \sum\left[\tfrac{1}{4}\eta_8^{ad}\left(m^{be}m^{cf} - m^{bf}m^{ce} + qm^{bcef}\right)\right] \\ &\quad -\frac{1}{16}\zeta^*\zeta\begin{vmatrix} \eta_8^{ad} & \eta_8^{ae} & \eta_8^{af} \\ \eta_8^{bd} & \eta_8^{be} & \eta_8^{bf} \\ \eta_8^{cd} & \eta_8^{ce} & \eta_8^{cf} \end{vmatrix} \end{aligned} \tag{46}$$

In (46) the summation sign indicates the sum of the nine terms obtained by independent cyclic interchange of a, b, c and of d, e, f.

Some interesting special cases of these identities which will prove useful in Section 6 are ($\kappa, \lambda = 0,1,2,3$):

$$m^{\kappa 4} m_{\kappa}{}^{4} = (m^{47})^2 + (m^{48})^2 \tag{47}$$

$$m^{\kappa 478} m_{\kappa}{}^{478} = -\tfrac{1}{4}\left[(m^{47})^2 + (m^{48})^2\right] \tag{48}$$

$$\mathrm{Re}(\zeta^* \zeta^{\kappa 478})\mathrm{Re}(\zeta^* \zeta_{\kappa}{}^{478}) = -\tfrac{1}{4}\zeta^*\zeta\left[(m^{47})^2 + (m^{48})^2\right] \tag{49}$$

$$m^{\kappa 4} m_{\kappa}{}^{478} = 0 \tag{50}$$

$$m^{\kappa 4}\mathrm{Re}(\zeta^* \zeta_{\kappa}{}^{478}) = 0 \tag{51}$$

$$m^{\kappa 478}\mathrm{Re}(\zeta^* \zeta_{\kappa}{}^{478}) = 0 \tag{52}$$

$$m^{\kappa 4}(m^{7}{}_{\kappa} + i m^{8}{}_{\kappa}) = (m^{47} + i m^{48})(m^{56} + i m^{78}) \tag{53}$$

$$m^{\kappa 478}(m^{7}{}_{\kappa} + i m^{8}{}_{\kappa}) = (m^{47} + i m^{48})(m^{5678} + \tfrac{1}{4} i q) \tag{54}$$

$$\mathrm{Re}(\zeta^* \zeta^{\kappa 478})(m^{7}{}_{\kappa} + i m^{8}{}_{\kappa}) = (m^{47} + i m^{48})\left[\mathrm{Re}(\zeta^* \zeta^{5678}) + \tfrac{1}{4} i \zeta^* \zeta\right] \tag{55}$$

$$\begin{aligned} m^{\kappa\lambda}(m^{47} + i m^{48}) &- (m^{7\kappa} + i m^{8\kappa}) m^{\lambda 4} + (m^{7\lambda} + i m^{8\lambda}) m^{\kappa 4} \\ &= q(m^{\kappa\lambda 47} + i m^{\kappa\lambda 48}) + \mathrm{Re}(\zeta^* \zeta^{\kappa\lambda 47}) + i\,\mathrm{Re}(\zeta^* \zeta^{\kappa\lambda 48}) \\ &= i\varepsilon^{\kappa\lambda\mu\nu}\left[(m^{7}{}_{\mu} + i m^{8}{}_{\mu}) m_{\nu}{}^{4} - \tfrac{1}{2}(m^{47} + i m^{48}) m_{\mu\nu}\right] \end{aligned} \tag{56}$$

As in previous sections a superscript 4 denotes the combination 5-6, thus $m^{47} = m^{57} - m^{67}$, $m^{\kappa\lambda 48} = m^{\kappa\lambda 58} - m^{\kappa\lambda 68}$, etc.

6. SPECULATIONS ON D^8

6.1. Introduction. Minkowski space M_4 has proved a very useful mathematical tool for modeling the macroscopic physical world in the absence of gravity. But on the microscopic scale the space-time continuum concept could well need modification, perhaps quantization in some sense. We certainly cannot apply our usual meter stick and clock operations to measure nuclear and subnuclear distances and times, which throws doubt on the meaning of the latter.

Rzewuski (1958) suggested that Minkowski space vectors should be derived from a more basic four-dimensional space of complex spinors. His ideas were given a firm basis in the twistor theory of Penrose and coworkers, where the substructure behind M_4 is a complex projective 3-space whose points are $O(2,4)$ spinors. These correspond to the null straight lines in M_4 in the way described in Section 4. A space-time event is given as the intersection of null lines, necessitating two or more twistors for its representation.

In n-twistor particle theory, $n \geqslant 2$, both the external and internal symmetries of massive particles can be modeled. For recent reviews see Rzewuski (1982) and Lukács *et al.* (1982).

A different approach is taken by Nash (1980, 1981a), who replaces M_4 as the basic manifold by what is in a rough sense a square root of M_4, a real 16-dimensional space constructed from pairs of real $O(3,3)$ spinors subject to certain constraints. Nash finds quadratic forms in the 16 spinor components which model the usual position, momentum, and angular momentum variables of a free particle of arbitrary mass and spin. The Poincaré transformation properties of the $O(3,3)$ spinors are given in terms of Penrose twistors. In a later paper Nash (1981b) modifies the twistor translation transformation law and adds to the pair of $O(3,3)$ spinors a real scalar. These 17 variables transform together according to a nonlinear "hyperspinor" transformation law and enable the motion of a charged particle in an external magnetic field to be modeled.

The speculations of this section have a similar motivation to the above work based on twistors, i.e., a desire to find a primitive entity from which M_4 is derived. We expand on the idea expressed in an earlier paper (Derrick, 1982) that the basic space is that of the D^8 spinors introduced in Section 5. Thus we postulate an eight-dimensional complex vector space whose coordinates ψ^A transform under Poincaré transformations and under charge conjugation according to D^8. We show in this section how particle variables in Minkowski space can be modeled in terms of the bilinear forms q, m^{ab}, m^{abcd}, the real linear combinations of $(\psi^A)^*\psi^B$ defined in Section 5.7.

6.2. Identification of Particle Variables. A particle of nonzero rest mass m has associated with it a number of Poincaré vectors and tensors. There is the position vector x^κ, linear momentum p^κ, angular momentum $j^{\kappa\lambda}$, spin angular momentum $s^{\kappa\lambda}$, and the Pauli–Lubanski spin vector w^κ. In the notation of Section 3, x^κ belongs to D^5, p^κ to $\operatorname{sgn}(L_0^0)D_0^4$, $(p^\kappa, j^{\kappa\lambda})$ to D^{10}, $s^{\kappa\lambda}$ to D_0^6, and w^κ to $(\det L)D_0^4$. These quantities are not all independent but are constrained by the identities

$$p^\kappa p_\kappa = m^2c^2 \tag{57}$$

$$p^\kappa w_\kappa = 0 \tag{58}$$

$$j^{\kappa\lambda} = x^\kappa p^\lambda - x^\lambda p^\kappa + s^{\kappa\lambda} \tag{59}$$

$$s^{\kappa\lambda} = (mc)^{-1}\varepsilon^{\kappa\lambda\mu\nu}w_\mu p_\nu \tag{60}$$

Our basic postulate is that x^κ, p^κ, $j^{\kappa\lambda}$, $s^{\kappa\lambda}$, w^κ can be constructed from the bilinear quantities q, m^{ab}, m^{abcd} formed from a D^8 spinor ψ^A. In making the identification we are guided by two constraints. Firstly, the identities (57)–(60) must be a consequence of the identities (43)–(56). Secondly, we must match the Poincaré transformation properties.

Let us assume that m^{47} and m^{48} are not both zero, and define the Poincaré scalar $N=[(m^{47})^2+(m^{48})^2]^{1/2}$. Then (47) shows that $m^{\kappa 4}/N$ is a timelike unit vector, which belongs to $\operatorname{sgn}(L_0^0)D_0^4$ according to Section 5.9. From the explicit form (24) one readily shows that it is future pointing. Comparison with (57) then suggests the identification

$$\begin{aligned} p^\kappa &= (mc)m^{\kappa 4}/N \\ &= \bar{\psi}P^\kappa\psi/N \end{aligned} \tag{61}$$

where we take $l=\hbar/(mc)$ in (13). We can regard N as a normalization denominator which ensures that ψ and $\lambda\psi$ yield the same p^κ for any complex multiplier λ.

The relation of p^κ to P^κ in (61) suggests that we define the "expectation value" of any 8×8 matrix $\mathscr{L}$ by $\langle\mathscr{L}\rangle=\bar{\psi}\mathscr{L}\psi/N$, ψ and $\lambda\psi$ yielding the same result. If $\mathscr{L}$ is self adjoint with respect to β, i.e., $\mathscr{L}^\dagger=\beta\mathscr{L}\beta$, then $\langle\mathscr{L}\rangle$ is real. Note that such expectation values can take any value in the range $\pm\infty$ because the denominator N can be as small as we like. Contrast this with the quantum mechanical averages $\psi^\dagger\mathscr{M}\psi/\psi^\dagger\psi$ which must lie between the greatest and least eigenvalues for a Hermitian matrix $\mathscr{M}$.

The form $p^\kappa=\langle P^\kappa\rangle$ of (61) now suggests that the particle angular momentum be identified similarly:

$$\begin{aligned} j^{\kappa\lambda} &= \langle J^{\kappa\lambda}\rangle \\ &= \hbar m^{\kappa\lambda}/N \end{aligned} \tag{62}$$

which is consistent with $(p^\kappa, j^{\kappa\lambda})$ belonging to D^{10}.

Finally we can satisfy (58)–(60) by the identifications

$$\begin{aligned} x^\kappa &= l\operatorname{Re}\left[(m^{7\kappa}+im^{8\kappa})/(m^{47}+im^{48})\right] \\ &= \left[\hbar/(mc)\right](m^{47}m^{7\kappa}+m^{48}m^{8\kappa})/N^2 \end{aligned} \tag{63}$$

$$\begin{aligned} w^\kappa &= -\hbar\left\{\operatorname{Im}\left[(m^{7\kappa}+im^{8\kappa})/(m^{47}+im^{48})\right]-m^{78}m^{\kappa 4}/N^2\right\} \\ &= \hbar\left[qm^{\kappa 478}+\operatorname{Re}(\zeta^*\zeta^{\kappa 478})\right]/N^2 \end{aligned} \tag{64}$$

$$s^{\kappa\lambda} = \hbar\left\{m^{47}\left[qm^{\kappa\lambda 47}+\operatorname{Re}(\zeta^*\zeta^{\kappa\lambda 47})\right]+m^{48}\left[qm^{\kappa\lambda 48}+\operatorname{Re}(\zeta^*\zeta^{\kappa\lambda 48})\right]\right\}/N^2 \tag{65}$$

This can be seen by dividing (56) and (53) by $(m^{47}+im^{48})$ and then taking the real and imaginary parts. As a check we note that the vectors given by (63), (64), (65) belong to D^5, $(\det L)D_0^4$, and D_0^6 as required. The square of the spin is $\frac{1}{2}s^{\kappa\lambda}s_{\kappa\lambda} = -w_\kappa w^\kappa = \frac{1}{4}\hbar^2(q^2+\zeta^*\zeta)/N^2$, where we have used (48), (49), and (52).

The particle vectors and tensors assigned by (61)–(65) are all invariant under charge conjugation. How should we interpret quantities like $q, m^{abcd}, \mathrm{Re}(\zeta), \mathrm{Re}(\zeta^{abcd}), \mathrm{Im}(\zeta^*\zeta^{abcd})$ which change sign on charge conjugation? Either of the scalars q, $\mathrm{Re}(\zeta)$ could be candidates for the charge. For the magnetic moment we need a D_0^6 tensor which reverses on charge conjugation, and there are many possibilities to choose from: $m^{\kappa\lambda 47}$, $\mathrm{Re}(\zeta^{\kappa\lambda 47})$, $(m^{47}m^{\kappa\lambda 47}+m^{48}m^{\kappa\lambda 48})/N^2$, $s^{\kappa\lambda}/q$, $\mathrm{Re}(\zeta^*\zeta^{\kappa\lambda 47})/q$, etc. Before making a definite assignment one needs to incorporate electromagnetic interactions into the theory. A tentative attempt in this direction is made in Section 6.4.

A simplified assignment of particle variables was suggested in an earlier paper (Derrick, 1982). There ψ was constrained to lie in the Poincaré invariant subspaces

$$m^{47}=1, \qquad m^{48}=0=m^{78}=\zeta \tag{66}$$

We then obtain

$$\begin{aligned} p^\kappa &= (mc)m^{\kappa 4} \\ j^{\kappa\lambda} &= \hbar m^{\kappa\lambda} \\ x^\kappa &= [\hbar/(mc)]\, m^{7\kappa} \\ w^\kappa &= -\hbar m^{8\kappa} = \hbar q m^{\kappa 478} \\ s^{\kappa\lambda} &= \hbar q m^{\kappa\lambda 47} \end{aligned} \tag{67}$$

In what follows we shall *not* assume the constraints (66) but retain the more general assignments (61)–(65), unless explicitly stated.

6.3. Equations of Motion. Along the trajectory of a free particle we have evolution in proper time s according to $(mc)\, dx^\kappa/ds = p^\kappa$, $dp^\kappa/ds = 0$, $dj^{\kappa\lambda}/ds = 0$, $dw^\kappa/ds = 0$. We can find equations of motion for $\psi^A(s)$ such that the averages (61)–(64) evolve in this way. A suitable Hamiltonian formulation with ψ^A and $\bar\psi_A = (\psi^B)^*\beta_{\dot B A}$ as conjugate variables is

$$\begin{aligned} H &= -mcN \\ i\hbar(d\psi^A/ds) &= \partial H/\partial\bar\psi_A \\ i\hbar(d\bar\psi_A/ds) &= -\partial H/\partial\psi^A \end{aligned} \tag{68}$$

(68) leads to

$$i\hbar(d\psi/ds)=\Xi\psi \tag{69}$$

where $\Xi=-mc(m^{47}M^{47}+m^{48}M^{48})/N$ is self-adjoint with respect to β. Multiplying (69) by the projection operators, $E_1, E_2=\frac{1}{2}(1\pm\tau_3)=\frac{1}{2}\mp iM^{56}$ and using $E_1\Xi=0=\Xi E_2$, $E_2\Xi=\Xi=\Xi E_1$ gives the alternative form

$$i\hbar d(E_1\psi)/ds=0$$

$$i\hbar d(E_2\psi)/ds=\Xi(E_1\psi) \tag{70}$$

As a consequence of (69) the forms $\bar{\psi}F\psi$ and $\psi^TCF\psi$ satisfy

$$i\hbar d(\bar{\psi}F\psi)/ds=\bar{\psi}[F,\Xi]\psi$$

$$i\hbar d(\psi^TCF\psi)/ds=\psi^TC[F,\Xi]\psi \tag{71}$$

Applying (71) we find that the particle variables do indeed satisfy the correct equations of motion. In addition to p^κ, $j^{\kappa\lambda}, w^\kappa$ further constants of motion are m^{47}, m^{48}, m^{78}, q, ζ, m^{5678}, ζ^{5678}, $m^{\kappa 478}$, $\zeta^{\kappa 478}$, $m^{\kappa\lambda 47}$, $m^{\kappa\lambda 48}$, $\zeta^{\kappa\lambda 47}$, $\zeta^{\kappa\lambda 48}$, $E_1\psi$.

Though (69) is nonlinear it is readily solved on account of $d\Xi/ds=0$, $\Xi^2=0$:

$$\psi(s)=(I-is\Xi/\hbar)\psi(0) \tag{72}$$

In (72) Ξ, being constant, may be evaluated using the $s=0$ value of the spinor, $\psi(0)$.

6.4. Motion in an Electromagnetic Field. In a constant external electromagnetic field $F^{\kappa\lambda}$ the trajectory of a particle of charge e is given by

$$dx^\kappa/ds=v^\kappa$$

$$dv^\kappa/ds=[e/(mc^2)]\,F^\kappa{}_\lambda v^\lambda \tag{73}$$

We shall assume that x^κ is derived from a D^8 spinor ψ according to (63) and $v^\kappa=m^{\kappa 4}/N$ in analogy with (61).

Integrating (73) gives $[e/(mc^2)]F^\kappa{}_\lambda x^\lambda-v^\kappa=\text{const}=a^\kappa$, say, so that (73) takes the form

$$dx^\kappa/ds=\omega^\kappa{}_\lambda x^\lambda-a^\kappa$$

$$dv^\kappa/ds=\omega^\kappa{}_\lambda v^\lambda \tag{74}$$

with $\omega_{\kappa\lambda}=[e/(mc^2)]F_{\kappa\lambda}$. Hence in proper time ds a coordinate frame attached to the particle undergoes a translation $a^\kappa\, ds$ and a Lorentz transformation $\omega_{\kappa\lambda}\, ds$. Since the matrices P^κ and $J^{\kappa\lambda}$ of (24) generate translations and Lorentz transformations, respectively, the change in ψ in proper time ds is

$$d\psi=\left(-ia_\kappa P^\kappa-\tfrac{1}{2}i\omega_{\kappa\lambda}J^{\kappa\lambda}\right)\psi\, ds/\hbar \tag{75}$$

Eliminating the constants a_κ and $\omega_{\kappa\lambda}$ in favor of v_κ and $F_{\kappa\lambda}$ then yields the equation of motion

$$i\hbar\, d\psi/ds=\left[-v_\kappa P^\kappa+eF_{\kappa\lambda}(J^{\kappa\lambda}-x^\kappa P^\lambda+x^\lambda P^\kappa)/(2mc^2)\right]\psi \tag{76}$$

In Appendix B Pauli–Fierz-like identities (77) are exploited to cast (76) into a form more like (69). The Pauli–Lubanski vector evolves as a consequence of (76) according to $dw^\kappa/ds=[e/(mc^2)]F^\kappa{}_\lambda w^\lambda$. When the magnetic moment is anomalous we can obtain the Bargmann–Michel–Telegdi equation for w^κ by replacing $F_{\kappa\lambda}$ in (76) by $\tfrac{1}{2}gF_{\kappa\lambda}+(\tfrac{1}{2}g-1)(v_\kappa F_{\lambda\mu}-v_\lambda F_{\kappa\mu})v^\mu$. This parallels the hyperspinor treatment of Nash (1981b).

6.5. Scope for Further Work. If the speculations of Section 6 are to form the basis of a viable theory then one needs a satisfactory treatment of the interrelated problems of (a) many-particle theory, (b) interactions, (c) quantization.

Concerning (a), at the crudest level one could represent an n-particle system by n different D^8 spinors $\psi_1^A, \psi_2^A, \ldots, \psi_n^A$, and generate equations of particle motion from some Poincaré invariant Hamiltonian formed from these variables. An alternative treatment might be based on a D^8 spinor of rank $n, \psi^{A_1A_2\cdots A_n}$, with the possibility of incorporating Bose and Fermi statistics through the symmetry properties of the indices.

With respect to problem (b), interactions, we can readily construct *ad hoc* theories for particular interactions along the lines of Section 6.4. At a more fundamental level one could seek a generalization of the D^8 geometry analogous to the transition from the special theory of relativity to the general theory. Here we have two types of metric, C_{AB} and $\beta_{\dot{A}B}$, which in a generalized theory might be replaced by functions of ψ^E and of $\psi^E, (\psi^F)^*$, respectively. Under a general analytic coordinate transformation $(\psi^A)'=$ function (ψ^B) in a local coordinate patch we would have the transformation laws $(C_{AB})'=C_{EF}J^E{}_AJ^F{}_B$ and $(\beta_{\dot{A}B})'=\beta_{\dot{E}F}(J^E{}_A)^*J^F{}_B$, where $J^E{}_A=\partial\psi^E/\partial(\psi^A)'$. One possibility would be to introduce an "achtbein" $u^{(A)}{}_E(\psi)$

transforming as $[u^{(A)}{}_E]' = u^{(A)}{}_B J^B{}_E$ with $C_{EF} = C^0_{AB} u^{(A)}{}_E u^{(B)}{}_F$ and $\beta_{\dot{E}F} = \beta^0_{\dot{A}B}[u^{(A)}{}_E]^* u^{(B)}{}_F$. C^0_{AB} and $\beta^0_{\dot{A}B}$ refer to the constant "flat space" values. One would then seek differential field equations for the $u^{(A)}{}_E$ which are invariant both under arbitrary coordinate changes $\psi^E \to (\psi^E)'$ and under D^8 transformations of the achtbein superscript (A).

Finally, the question (c) of quantisation could be approached via the Wigner operators $M^{ab}_{OP} = -(M^{ab})^E{}_F \psi^F \partial/\partial\psi^E$ which satisfy the same $SO(2,6)$ commutators as the matrices M^{ab}. The problem is to define a suitable linear vector space of analytic functions of ψ^A on which the M^{ab}_{OP} operate. We need to define a scalar product in this space in such a way that $P^\kappa_{OP} = \hbar M^{\kappa 4}_{OP}/l$ and $J^{\kappa\lambda}_{OP} = \hbar M^{\kappa\lambda}_{OP}$ become Hermitian operators on a Hilbert space. If this can be done then the simultaneous eigenstates of P^κ_{OP} would be interpreted as energy-momentum eigenstates.

These questions will be addressed in subsequent publications.

APPENDIX A: EXPLICIT VALUES OF M^{ab}, M^{abcd}

In terms of the three copies of the Pauli matrices $\boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\tau}$ introduced in Section 5.2, the $SO(2,6)$ generators M^{ab} are

$$(M^{23}, M^{31}, M^{12}) = \tfrac{1}{2}\boldsymbol{\sigma}$$

$$(M^{01}, M^{02}, M^{03}) = \tfrac{1}{2} i\rho_1 \boldsymbol{\sigma}$$

$$(M^{05}; M^{15}, M^{25}, M^{35}) = \tfrac{1}{2}\tau_1(i\rho_2; \rho_3\boldsymbol{\sigma})$$

$$(M^{06}; M^{16}, M^{26}, M^{36}) = \tfrac{1}{2} i\tau_2(i\rho_2; \rho_3\boldsymbol{\sigma})$$

$$(M^{07}; M^{17}, M^{27}, M^{37}) = -\tfrac{1}{2}\tau_3(i\rho_2; \rho_3\boldsymbol{\sigma})$$

$$(M^{08}; M^{18}, M^{28}, M^{38}) = \tfrac{1}{2}(-i\rho_3; \rho_2\boldsymbol{\sigma})$$

$$M^{56} = \tfrac{1}{2} i\tau_3, \qquad M^{57} = \tfrac{1}{2}\tau_2, \qquad M^{58} = -\tfrac{1}{2}\tau_1\rho_1$$

$$M^{67} = -\tfrac{1}{2} i\tau_1, \qquad M^{68} = -\tfrac{1}{2} i\tau_2\rho_1, \qquad M^{78} = \tfrac{1}{2}\tau_3\rho_1$$

The products $M^{abcd} = M^{ab}M^{cd} \equiv -(1/24)\varepsilon^{abcdefgh}M_{efgh}$ $(a, b, c, d$ all different) are given below. To save writing we omit the symbol M and abbreviate M^{abcd} to *abcd*:

$$0123 = 5678 = \tfrac{1}{4} i \rho_1$$

$$(1235; 0235, 0315, 0125) = -(0678; 1678, 2678, 3678)$$

$$= \tfrac{1}{4} \tau_1 (\rho_3; i\rho_2 \sigma)$$

$$(1236; 0236, 0316, 0126) = -(0578; 1578, 2578, 3578)$$

$$= \tfrac{1}{4} i \tau_2 (\rho_3; i\rho_2 \sigma)$$

$$(1237; 0237, 0317, 0127) = -(0568; 1568, 2568, 3568)$$

$$= -\tfrac{1}{4} \tau_3 (\rho_3; i\rho_2 \sigma)$$

$$(1238; 0238, 0318, 0128) = (0567; 1567, 2567, 3567)$$

$$= \tfrac{1}{4} (\rho_2; -i\rho_3 \sigma)$$

$$(2356, 3156, 1256) = (0178, 0278, 0378)$$

$$= \tfrac{1}{4} i \tau_3 \sigma$$

$$(0156, 0256, 0356) = -(2378, 3178, 1278)$$

$$= -\tfrac{1}{4} \tau_3 \rho_1 \sigma$$

$$(2357, 3157, 1257) = (0168, 0268, 0368)$$

$$= \tfrac{1}{4} \tau_2 \sigma$$

$$(0157, 0257, 0357) = -(2368, 3168, 1268)$$

$$= \tfrac{1}{4} i \tau_2 \rho_1 \sigma$$

$$(2358, 3158, 1258) = -(0167, 0267, 0367)$$

$$= -\tfrac{1}{4} \tau_1 \rho_1 \sigma$$

$$(0158, 0258, 0358) = (2367, 3167, 1267)$$

$$= -\tfrac{1}{4} i \tau_1 \sigma$$

Taken together the M^{ab}, M^{abcd} are infinitesimal generators of $SU(4,4)$. The commutators $[M^{ab}, M^{cd}]$ are those of the $SO(2,6)$ subgroup and are given by (23). The remaining commutators are

$$[M^{ab}, M^{cdef}] = i\left(\eta_8^{bc}M^{adef} + \eta_8^{bd}M^{caef} + \eta_8^{be}M^{cdaf} + \eta_8^{bf}M^{cdea} - a \rightleftharpoons b\right)$$

$$[M^{abcd}, M^{efgh}] = \tfrac{1}{8}i\left(\eta_8^{ae}\varepsilon^{bcdfghjk}M_{jk} + 15 \text{ similar terms}\right)$$

$$-\tfrac{1}{4}i\begin{vmatrix} \eta_8^{ae} & \eta_8^{af} & \eta_8^{ag} \\ \eta_8^{be} & \eta_8^{bf} & \eta_8^{bg} \\ \eta_8^{ce} & \eta_8^{cf} & \eta_8^{cg} \end{vmatrix} M^{dh} + 15 \text{ similar terms}$$

The omitted terms above are those needed to antisymmetrize the right-hand side with respect to both a, b, c, d and e, f, g, h.

APPENDIX B: PROOF OF (43) TO (46)

Direct proof of these identities would require an inordinate amount of writing. Instead we exploit their $SO(2,6)$ tensor form, which implies that if they are true for a particular D^8 spinor ϕ they are also true for any spinor ψ derivable from ϕ by an $SO(2,6)$ transformation $\exp(\eta_8\Omega)$, viz. $\psi = \exp(-\frac{1}{2}i\Omega_{ab}M^{ab})\phi$. Here $[\Omega_{ab}]$ is an arbitrary antisymmetric 8×8 matrix. Our method of proof is to verify the identities for a spinor ϕ of particularly simple form, then show that a sufficiently dense set of spinors ψ can be transformed into ϕ for the identities to be true in general.

The special spinor adopted is $\phi = (a_1, a_2, 0, 0, 0, 0, a_1, -a_2)^T$, where a_1, a_2 are complex numbers of which at least one does not vanish. The nonzero values of $m^{ab} = \bar{\phi}M^{ab}\phi$ are $m^{37} = m^{58} = -m^{12} = -m^{06} = |a_1|^2 + |a_2|^2$, while $q = \bar{\phi}\phi = 2(|a_2|^2 - |a_1|^2)$ and $\zeta = \phi^T C\phi = 4a_1a_2$. Similar simple expressions apply to $m^{abcd} = \bar{\phi}M^{abcd}\phi$ and $\zeta^{abcd} = \phi^T CM^{abcd}\phi$, for each of which only 11 fail to vanish. It is now a trivial matter to verify (43)–(46) for this simple case.

We now have to prove the following:

> *Theorem.* Given any two D^8 spinors ψ, ϕ which have the same β-norms $\bar{\psi}\psi = \bar{\phi}\phi = q$ and C-norms $\psi^T C\psi = \phi^T C\phi = \zeta$, then if q, ζ are not both zero we can transform ϕ into ψ by a D^8 transformation, i.e., $D = \exp(-\frac{1}{2}i\Omega_{ab}M^{ab})$ exists such that $\psi = D\phi$.

Proof. Suppose first that $q \neq 0$. The four vectors $\phi, \mathscr{C}\phi, \psi, \mathscr{C}\psi$ have the β-norms $\bar{\phi}\phi = q$, $\overline{\mathscr{C}\phi}\mathscr{C}\phi = -q$, $\bar{\psi}\psi = q$, $\overline{\mathscr{C}\psi}\mathscr{C}\psi = -q$. Since β has eigenvalues $1, 1, 1, 1, -1, -1, -1, -1$ we can find a vector χ of β-norm $\bar{\chi}\chi = q$ orthogonal to the four vectors: $\bar{\chi}\phi = 0$, $\bar{\chi}\mathscr{C}\phi = 0$, $\bar{\chi}\psi = 0$, $\bar{\chi}\mathscr{C}\psi = 0$. $\mathscr{C}\chi$ is orthogonal to the same vectors and has β-norm $-q$. Now we can always choose χ such that $\chi^T C\chi = \zeta$. For if $\chi^T C\chi = 0$ then $\chi' = \chi\cosh\theta + \mathscr{C}\chi(\zeta/|\zeta|)\sinh\theta$ with $\sinh 2\theta = |\zeta|/q$ has the desired property. Otherwise, if $\chi^T C\chi = \nu \neq 0$ we first construct $\chi_0 = [\chi\cosh\theta_0 - \mathscr{C}\chi(\nu/|\nu|)\sinh\theta_0]/\cosh 2\theta_0$ with $\sinh 2\theta_0 = |\nu|/q$, which satisfies $\bar{\chi}_0\chi_0 = q$, $\chi_0^T C\chi_0 = 0$, then proceed as before.

Let us assume then that $\chi^T C\chi = \zeta$. Define

$$\Delta_1 = iz\Big\{q\big[\phi\bar{\chi} - \chi\bar{\phi} - (\mathscr{C}\chi)\phi^T C + (\mathscr{C}\phi)\chi^T C$$

$$+ \zeta^*(\phi\chi^T - \chi\phi^T)C + \zeta\big[(\mathscr{C}\chi)\bar{\phi} - (\mathscr{C}\phi)\bar{\chi}\big]\Big\}$$

where z is a real parameter. We have $\Delta_1^\dagger = \beta\Delta_1\beta$ and $\Delta_1^T = -C\Delta_1 C$, which means that Δ_1 is a linear combination of M^{ab} with real coefficients, $\Delta_1 = \frac{1}{2}\Omega_{1ab}M^{ab}$ with $\Omega_{1ab} = \frac{1}{2}\mathrm{Tr}(M_{ab}\Delta_1)$. By explicit calculation we find $[\exp(i\Delta_1)]\phi = \phi\cos\alpha + \chi\sin\alpha$ with $\alpha = z(q^2 + |\zeta|^2)$. Choose $\alpha = \frac{1}{2}\pi$ so that $[\exp(i\Delta_1)]\phi = \chi$. A Δ_2 of similar form exists for which $[\exp(i\Delta_2)]\psi = \chi$, whence $D = \exp(-i\Delta_2)\exp(i\Delta_1)$ is a D^8 transformation effecting the mapping $\psi = D\phi$. Because of the group representation property $D = \exp(-\frac{1}{2}i\Omega_{ab}M^{ab})$ for some Ω_{ab}.

If we have $q = 0$, $\zeta \neq 0$ the argument proceeds as before with the roles of the β-norm and the C-norm interchanged. This completes the proof of the theorem.

Hence we have proved the identities provided q and ζ are not both zero. However because of the continuity of the bilinear forms with respect to ψ the identities remain true in the limit $q = \zeta = 0$ so that they are true for all ψ.

Further Pauli–Fierz-type identities can be derived by the technique of transforming ψ to the special form ϕ. Two interesting identities which lead to an alternative form of (76) are

$$\left(m^{ae}M_e{}^b - \tfrac{1}{2}im^{ab}\right)\psi = -\left(\tfrac{1}{2}iM_{ab} + \tfrac{1}{4}\eta_8^{ab}\right)(q\psi - \zeta\mathscr{C}\psi)$$

$$\left(m^{ab}M^{cd} + m^{ac}M^{db} + m^{ad}M^{bc} - 2im^a{}_f M^{bcdf}\right)\psi$$

$$= \left[M^{abcd} - \tfrac{1}{2}i\left(\eta_8^{ab}M^{cd} + \eta_8^{ac}M^{db}\right.\right.$$

$$\left.\left. + \eta_8^{ad}M^{bc}\right)\right](q\psi - \zeta\mathscr{C}\psi) \qquad (77)$$

With the aid of (77) the terms on the right-hand side of (76) have the alternative forms

$$v_\kappa P^\kappa \psi = mc(m^{47}M^{47} + m^{48}M^{48})\psi/N$$

$$(J^{\kappa\lambda} - x^\kappa P^\lambda + x^\lambda P^\kappa)\psi/\hbar = 2i\Big[(mc/\hbar)x_\mu M^{\kappa\lambda\mu 4} - M^{\kappa\lambda 56} + m^{78}(m^{47}M^{\kappa\lambda 48} - m^{48}M^{\kappa\lambda 47})/N^2\Big]\psi + (m^{47}M^{\kappa\lambda 47} + m^{48}M^{\kappa\lambda 48})(q\psi - \zeta\mathscr{C}\psi)/N^2$$

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